

Coulomb gas integrals for commuting SLEs: Schramm's formula and Green's function

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Abstract

We use methods developed in conformal field theory to produce martingale observables for systems of commuting multiple SLE curves. In the case of two curves started from distinct points and growing towards infinity, we use these observables to determine rigorously and explicitly the Green's function and Schramm's formula. As corollaries, we obtain proofs of “fusion” formulas, some of which have been predicted in the physics literature. Our approach does not need *a priori* information on the regularity of the SLE observables, but does require a detailed analysis of the regularity and asymptotics of certain Coulomb gas contour integrals. These integrals are natural generalizations of the classic hypergeometric functions and are interesting in their own right. We indicate a method for computing the relevant asymptotics of these integrals to all orders.

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1 Introduction

Schramm-Loewner evolution (SLE) processes are universal lattice size scaling limits of interfaces in critical planar lattice models. By the Markovian property of SLE, geometric observables determine martingales with respect to the natural SLE filtration. Such *martingale observables* satisfy differential equations, which can sometimes be used to find explicit formulas for the observables, or be used as a basis for estimates, see for instance [35, 9]. A related problem is that of constructing martingales carrying some specific geometric information about the SLE process.

The differential equations are usually derived using Itô calculus, assuming sufficient regularity to apply Itô's formula. If a solution with suitable boundary values can be found, one way to proceed is to perform a probabilistic argument using the solution's boundary behavior to show that it actually represents the desired quantity. In the simplest cases the differential equation is an ODE, but generically it is a semi-elliptic PDE in several variables and it is difficult to find solutions with desired boundary data. (But see e.g. [15, 25] for cases when solutions are available.) Depending on the amount of available information, non-trivial regularity issues may need to be resolved when analyzing the PDEs, see, e.g. [16]. Seeking ways to construct solutions and methods for extracting information from them therefore seems worthwhile.

The basic PDEs that arise in SLE theory also arise in conformal field theory (CFT), see e.g. [8, 10, 5, 16]. On the other hand, conformal field theory provides ideas and methods for how to systematically construct solution candidates, see e.g. [6, 7, 23]. We will make use of the Coulomb gas framework of [23] which models the CFTs using certain Gaussian free field correlation functions. Very briefly, CFT correlations involving special fields inserted on the boundary give rise to SLE martingales and thereby PDE solutions. By making additional, carefully chosen, field insertions, the scaling behavior at the insertion points can in some cases be prescribed. In this way, using a “calculus of scaling dimensions”, many SLE martingale observables were recovered in [23] as CFT correlation functions.

The purpose of this paper is to construct one-point martingale observables (equivalently, to solve the corresponding PDEs) related to specific geometric information for multiple commuting SLEs by exploiting ideas from CFT. In the process we will suggest an approach for rigorously deriving at least some natural observables in this setting. The argument proceeds through three steps:

- (1) The first step generates a non-rigorous prediction for the observable by using a screening argument based on ideas from CFT [13, 23, 3]. (See also [12, 25] and the references in the latter.) The prediction is expressed as a contour integral with an explicit integrand. We call these integrals *screening integrals* or *Coulomb gas integrals*, see Section 4. The main difficulty is to choose the appropriate integration contour, but this choice can be simplified by considering appropriate limits.
- (2) The second step is to prove that the prediction from Step 1 satisfies the correct boundary conditions. This technical step involves the computation of somewhat complicated integral asymptotics, but we indicate an approach for computing such asymptotics in Section 10.

- (3) The last step is to use probabilistic methods together with the estimates of Step 2 to rigorously establish that the prediction from Step 1 gives the correct quantity.

We analyze two examples in detail. Both examples involve two curves aiming for ∞ with one marked point in the interior, but we will indicate how the arguments can be generalized to more complicated configurations.

The first example concerns the probability that the system of SLEs passes to the right of a given interior point; that is, the analog of Schramm’s formula [35]. This probability obviously depends only on the behavior of the leftmost curve. (So it is really an $\text{SLE}_\kappa(2)$ observable.) The main difficulty in this case lies in implementing Steps 1 and 2.

The second example concerns the limiting renormalized probability that the system of SLEs passes near a given point, that is, the Green’s function. We first give a proof of the non-trivial fact that the commuting Green’s function actually exists as a limit. The main step is to verify existence in the case when only one of the two curves grows. (More precisely, we prove the existence of the $\text{SLE}_\kappa(\rho)$ Green’s function, where ρ is in an interval and the force point is on the boundary.) The proof gives a representation formula in terms of an expectation for two-sided radial SLE stopped at its target point; this is similar to the main result of [2]. In Step 1, the prediction is made by taking a linear combination of screening integrals to cancel a leading order term which has the wrong asymptotics (thereby matching the asymptotics we expect). Then, in Step 2, we carefully analyze the candidate solution and estimate its boundary behavior. Lastly, given the estimates from Step 2, we show that the candidate observable enjoys the same probabilistic representation as the Green’s function defined as a limit – so they must agree.

By letting the seed points of the SLEs collapse, we obtain rigorous proofs of *fusion* formulas as corollaries. One can verify *a posteriori* that these limiting one-point observables satisfy specific third-order ODEs that can also be obtained from the so-called fusion rules of conformal field theory, see e.g. [12]. In fact, in the case of the Schramm probability, the formulas we prove here were predicted using fusion in [20]. The formulas for the fused Green’s functions appear to be new. The interpretation of fusion in SLE theory as successive merging of seeds, and the highly non-trivial fact that fused one-point observables satisfy higher order ODEs, was rigorously established in [16]. The ODEs for the Schramm formula for several fused paths were derived rigorously in [16] and the two-path formula in the special case $\kappa = 8/3$ (also allowing for two interior points) was established in [4].

As can be gathered from the length of the paper, there are many details to handle. In particular, the asymptotic analysis of the screening integrals is quite involved. These integrals are natural generalizations of hypergeometric functions and are interesting in their own right. Similar generalized hypergeometric functions have been considered before in related contexts, see e.g. [13, 15, 25, 26, 22]. However, we have not been able to find the required analytic estimates in the literature. In Section 10, we propose a method for establishing the asymptotics of these integrals to all orders in a rather general setting.

1.1 Outline of the paper

The main results of the paper are stated in Section 2, while we review some aspects of multiple SLE_κ and $\text{SLE}_\kappa(\rho)$ processes in Section 3.

In Section 4, we review and use ideas from conformal field theory to generate predictions for Schramm's formula and Green's function for commuting SLE with two curves growing toward infinity as Coulomb gas integrals.

In Section 5, we prove rigorously that the predicted Schramm's formula indeed gives the probability that a given point lies to the left of both curves. The proof relies on a number of technical asymptotic estimates; proofs of these estimates are collected in Appendix A.

In Section 6, we give a rigorous proof that the predicted Green's function equals the renormalized probability that the system passes near a given point. The proof relies both on pure SLE estimates (established in Section 6) and on asymptotic estimates for contour integrals (established in Section 10 and Appendix B).

In Section 7, we prove a lemma which expresses the fact that it is very unlikely that both curves in a commuting system get near a given point.

In Section 8, we consider the special case of two fused curves, i.e., the case when both curves in the commuting system start at the same point. In the case of Schramm's formula, this provides rigorous proofs of some predictions due to Gamsa and Cardy for Schramm's formula presented in [20].

In Section 9, we derive formulas for the Green's function in the special case when $8/\kappa$ is an integer.

In Section 10, we suggest an approach for computing certain asymptotics to all orders for a class of contour integrals which generalize the classic hypergeometric functions. The proofs in Appendix B are an application of this approach to the contour integral relevant for the Green's function.

In Section 11, we discuss our results from the point of view of differential equations.

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2 Main results

Before stating the main results, we review the definition of a system of two commuting SLE paths $\{\gamma_j\}_1^2$ in the upper half-plane $\mathbb{H} := \{\text{Im } z > 0\}$ growing toward infinity.

Let $0 < \kappa < 8$. Let $(\xi^1, \xi^2) \in \mathbb{R}^2$ with $\xi^1 \neq \xi^2$. The Loewner equation corresponding to two growing curves is

$$dg_t(z) = \frac{\lambda_1(t)dt}{g_t(z) - \xi_t^1} + \frac{\lambda_2(t)dt}{g_t(z) - \xi_t^2}, \quad g_0(z) = z, \quad (2.1)$$

where ξ_t^1 and ξ_t^2 , $t \geq 0$, are the driving terms for the two curves and the growth speeds $\lambda_j(t)$ satisfy $\lambda_j(t) \geq 0$. The solution of (2.1) is a family of conformal maps $(g_t(z))_{t \geq 0}$ called the Loewner chain of (ξ_t^1, ξ_t^2) . The *system of two commuting SLEs started from (ξ^1, ξ^2)* is obtained by taking ξ_t^1 and ξ_t^2 as solutions of the system

$$\begin{cases} d\xi_t^1 = \frac{\lambda_1(t) + \lambda_2(t)}{\xi_t^1 - \xi_t^2} dt + \sqrt{\frac{\kappa}{2} \lambda_1(t)} dB_t^1, & \xi_0^1 = \xi^1, \\ d\xi_t^2 = \frac{\lambda_1(t) + \lambda_2(t)}{\xi_t^2 - \xi_t^1} dt + \sqrt{\frac{\kappa}{2} \lambda_2(t)} dB_t^2, & \xi_0^2 = \xi^2, \end{cases} \quad (2.2)$$

where B_t^1 and B_t^2 are independent standard Brownian motions with respect to some measure $\mathbf{P} = \mathbf{P}_{\xi^1, \xi^2}$. The paths are defined by

$$\gamma_j(t) = \lim_{y \downarrow 0} g_t^{-1}(\xi_t^j + iy), \quad \gamma_{j,t} := \gamma_j[0, t], \quad j = 1, 2. \quad (2.3)$$

For $j = 1, 2$, $\gamma_{j,\infty}$ is a continuous curve connecting ξ^j with ∞ in \mathbb{H} . Given $z \in \mathbb{C} \setminus (\gamma_{1,\infty} \cup \gamma_{2,\infty})$, we let $\Upsilon_\infty(z)$ denote 1/2 times the conformal radius of $\mathbb{H} \setminus (\gamma_{1,\infty} \cup \gamma_{2,\infty})$ seen from z . It can be shown that the system (2.1) is commuting in the sense that the order in which the two curves are grown does not matter [15]. Since our theorems only concern the distribution of the fully grown curves $\gamma_{1,\infty}$ and $\gamma_{2,\infty}$, the choice of growth speeds is irrelevant.

Let us also recall the definition of (chordal) $\text{SLE}_\kappa(\rho)$ for a single path γ_1 in \mathbb{H} growing toward infinity. Let $\rho \in \mathbb{R}$ and let $(\xi^1, \xi^2) \in \mathbb{R}^2$ with $\xi^1 \neq \xi^2$. Let W_t be a standard Brownian motion with respect to some measure \mathbf{P}^ρ . Then *$\text{SLE}_\kappa(\rho)$ started from (ξ^1, ξ^2)* is defined by the equations

$$\begin{aligned} \partial_t g_t(z) &= \frac{2/\kappa}{g_t(z) - \xi_t^1}, \quad g_0(z) = z, \\ d\xi_t^1 &= \frac{\rho/\kappa}{\xi_t^1 - g_t(\xi^2)} dt + dW_t, \quad \xi_0^1 = \xi^1. \end{aligned}$$

When referring to $\text{SLE}_\kappa(\rho)$ started from (ξ^1, ξ^2) , we always assume that the curve starts from the first point of the tuple (ξ^1, ξ^2) while the second point (in this case ξ^2) is the force point. The $\text{SLE}_\kappa(\rho)$ path $\gamma_1(t)$ is defined as in (2.3). Given $z \in \mathbb{C} \setminus \gamma_{1,\infty}$, we let $\Upsilon_\infty(z)$ denote 1/2 times the conformal radius of $\mathbb{H} \setminus \gamma_{1,\infty}$ seen from z .

2.1 Schramm's formula

Our first result provides an explicit expression for the probability that an $\text{SLE}_\kappa(2)$ path passes to the right of a given point. The probability is expressed in terms of the function $\mathcal{M}(z, \xi)$ defined for $z \in \mathbb{H}$ and $\xi > 0$ by

$$\begin{aligned} \mathcal{M}(z, \xi) &= y^{\alpha-2} z^{-\frac{\alpha}{2}} (z - \xi)^{-\frac{\alpha}{2}} \bar{z}^{1-\frac{\alpha}{2}} (\bar{z} - \xi)^{1-\frac{\alpha}{2}} \\ &\quad \times \int_{\bar{z}}^z (u - z)^\alpha (u - \bar{z})^{\alpha-2} u^{-\frac{\alpha}{2}} (u - \xi)^{-\frac{\alpha}{2}} du, \end{aligned} \quad (2.4)$$

where $\alpha = 8/\kappa > 1$ and the integration contour from \bar{z} to z passes to the right of ξ , see Figure 1.

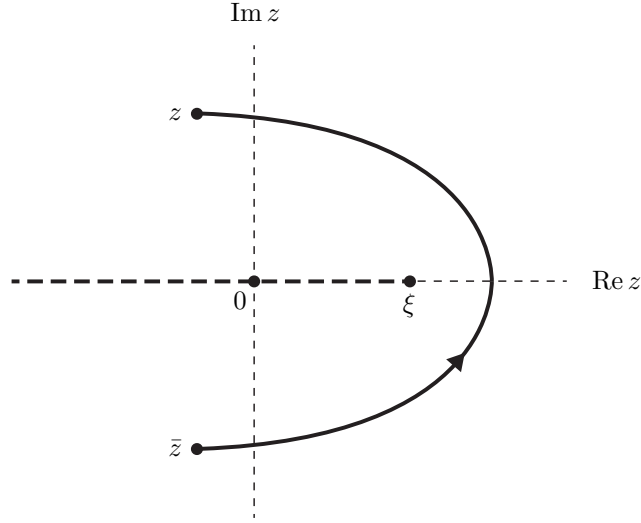


Figure 1. The integration contour used in the definition (2.4) of $\mathcal{M}(z, \xi)$ is a path from \bar{z} to z which passes to the right of ξ .

Theorem 2.1 (Schramm's formula for $\text{SLE}_\kappa(2)$). *Let $0 < \kappa < 8$. Let $\xi > 0$ and consider chordal $\text{SLE}_\kappa(2)$ started from $(0, \xi)$. Then the probability $P(z, \xi)$ that a given point $z = x + iy \in \mathbb{H}$ lies to the left of the curve is given by*

$$P(z, \xi) = \frac{1}{c_\alpha} \int_x^\infty \text{Re } \mathcal{M}(x' + iy, \xi) dx', \quad z \in \mathbb{H}, \quad \xi > 0, \quad (2.5)$$

where the normalization constant $c_\alpha \in \mathbb{R}$ is defined by

$$c_\alpha = -\frac{2\pi^{3/2}\Gamma\left(\frac{\alpha-1}{2}\right)\Gamma\left(\frac{3\alpha}{2}-1\right)}{\Gamma\left(\frac{\alpha}{2}\right)^2\Gamma(\alpha)}. \quad (2.6)$$

The proof of Theorem 2.1 will be given in Section 5. The form of the definition (2.5) of $P(z, \xi)$ is motivated by the CFT and screening considerations of Section 4.

A point $z \in \mathbb{H}$ lies to the left of both curves in a commuting system iff it lies to the left of the leftmost curve. Since each of the two curves of a commuting process has the distribution of an $\text{SLE}_\kappa(2)$ (see Section 3.1.2), Theorem 2.1 immediately implies the following result for commuting SLE.

Corollary 2.2 (Schramm's formula for two commuting SLEs). *Let $0 < \kappa < 8$. Let $\xi > 0$ and consider a system of two commuting SLE_κ curves in \mathbb{H} started from $(0, \xi)$ and growing toward infinity. Then the probability $P(z, \xi)$ that a given point $z = x + iy \in \mathbb{H}$ lies to the left of both curves is given by (2.5).*

Corollary 2.2 together with translation invariance immediately yields an expression for the probability that a point z lies to the left of a system of two SLEs started from two arbitrary points (ξ_1, ξ_2) in \mathbb{R} . The probabilities that z lies to the right of or between

the two curves then follow by symmetry. For completeness, we formulate this as another corollary.

Corollary 2.3. *Let $0 < \kappa < 8$. Suppose $-\infty < \xi^1 < \xi^2 < \infty$ and consider a system of two commuting SLE_κ curves in \mathbb{H} started from (ξ^1, ξ^2) and growing toward infinity. Let $P(z, \xi)$ denote the function in (2.5). Then the probability $P_{\text{left}}(z, \xi^1, \xi^2)$ that a given point $z = x + iy \in \mathbb{H}$ lies to the left of both curves is given by*

$$P_{\text{left}}(z, \xi^1, \xi^2) = P(z - \xi^1, \xi^2 - \xi^1);$$

the probability $P_{\text{right}}(z, \xi^1, \xi^2)$ that a point $z \in \mathbb{H}$ lies to the right of both curves is

$$P_{\text{right}}(z, \xi^1, \xi^2) = P(-\bar{z} + \xi^2, \xi^2 - \xi^1);$$

and the probability $P_{\text{middle}}(z, \xi^1, \xi^2)$ that z lies between the two curves is given by

$$P_{\text{middle}}(z, \xi^1, \xi^2) = 1 - P_{\text{left}}(z, \xi^1, \xi^2) - P_{\text{right}}(z, \xi^1, \xi^2).$$

By letting $\xi \rightarrow 0+$, we obtain proofs of formulas for “fused” paths. This gives a proof of the predictions of [20] where these formulas were derived by solving a third order ODE obtained from so-called fusion rules. We note that even given the explicit predictions of [20], it is not clear how to proceed to verify them rigorously. Indeed, as soon the evolution starts, the tips of the curves are separated and the system leaves the fused state. However, [16] provides a different rigorous approach by exploiting the hypoellipticity of the PDEs to show that the fused observables satisfy the higher order ODEs. The formula in the special case $\kappa = 8/3$ was proved in [4] using Cardy and Simmons’ prediction [11] for a two-point Schramm formula. See Theorem 8.1 for a complete statement and details.

Corollary 2.4. *Let $0 < \kappa < 8$ and define $P_{\text{fusion}}(z) = \lim_{\xi \downarrow 0} P(z, \xi)$, where $P(z, \xi)$ is as in (2.5). Then*

$$P_{\text{fusion}}(z) = \frac{\Gamma(\frac{\alpha}{2})\Gamma(\alpha)}{2^{2-\alpha}\pi\Gamma(\frac{3\alpha}{2}-1)} \int_{\frac{x}{y}}^{\infty} S(t') dt',$$

where the real-valued function $S(t)$ is defined by

$$S(t) = (1+t^2)^{1-\alpha} \left\{ {}_2F_1\left(\frac{1}{2} + \frac{\alpha}{2}, 1 - \frac{\alpha}{2}, \frac{1}{2}; -t^2\right) - \frac{2\Gamma(1 + \frac{\alpha}{2})\Gamma(\frac{\alpha}{2})t}{\Gamma(\frac{1}{2} + \frac{\alpha}{2})\Gamma(-\frac{1}{2} + \frac{\alpha}{2})} {}_2F_1\left(1 + \frac{\alpha}{2}, \frac{3}{2} - \frac{\alpha}{2}, \frac{3}{2}; -t^2\right) \right\}, \quad t \in \mathbb{R}.$$

2.2 Green’s function

Our second main result provides an explicit expression for the Green’s function for $SLE_\kappa(2)$.

Let $\alpha = 8/\kappa$. Define the function $I(z, \xi^1, \xi^2)$ for $z \in \mathbb{H}$ and $-\infty < \xi^1 < \xi^2 < \infty$ by

$$I(z, \xi^1, \xi^2) = \int_A^{(z+, \xi^2+, z-, \xi^2-)} (u - z)^{\alpha-1} (u - \bar{z})^{\alpha-1} (u - \xi^1)^{-\frac{\alpha}{2}} (\xi^2 - u)^{-\frac{\alpha}{2}} du, \quad (2.7)$$

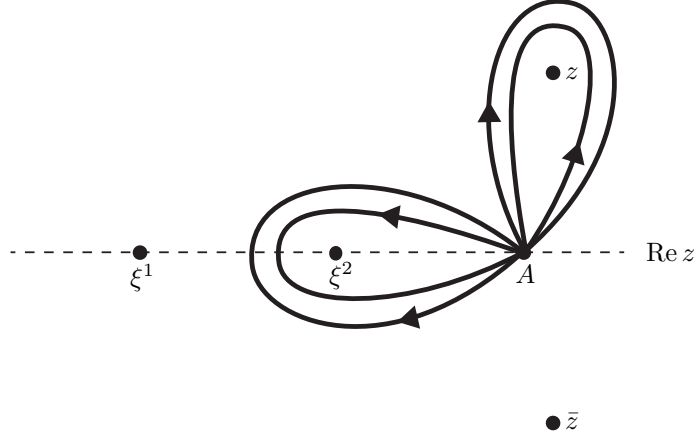


Figure 2. The Pochhammer integration contour in (2.7) is the composition of four loops based at the point $A > \xi_2$.

where $A > \xi^2$ is a basepoint and the Pochhammer integration contour is displayed in Figure 2. More precisely, the integration contour begins at the base point A , encircles the point z once in the positive sense, returns to A , encircles ξ^2 once in the positive sense, returns to A , and so on. The points \bar{z} and ξ^1 are exterior to all loops. The factors in the integrand take their principal values at the starting point and are then analytically continued along the contour.

For $\alpha \in (1, \infty) \setminus \mathbb{Z}$, we define the function $\mathcal{G}(z, \xi^1, \xi^2)$ by

$$\mathcal{G}(z, \xi^1, \xi^2) = \frac{1}{\hat{c}} y^{\alpha + \frac{1}{\alpha} - 2} |z - \xi^1|^{1-\alpha} |z - \xi^2|^{1-\alpha} \operatorname{Im} (e^{-i\pi\alpha} I(z, \xi^1, \xi^2)), \quad z \in \mathbb{H}, \quad \xi^1 < \xi^2, \quad (2.8)$$

where the constant $\hat{c} = \hat{c}(\kappa)$ is given by

$$\hat{c} = \frac{4 \sin^2 \left(\frac{\pi\alpha}{2} \right) \sin(\pi\alpha) \Gamma \left(1 - \frac{\alpha}{2} \right) \Gamma \left(\frac{3\alpha}{2} - 1 \right)}{\Gamma(\alpha)} \quad \text{with} \quad \alpha = \frac{8}{\kappa}. \quad (2.9)$$

We extend this definition of $\mathcal{G}(z, \xi^1, \xi^2)$ to all $\alpha > 1$ by continuity. The following lemma shows that even though \hat{c} vanishes as α approaches an integer, the function $\mathcal{G}(z, \xi^1, \xi^2)$ has a continuous extension to integer values of α .

Lemma 2.5. *For each $z \in \mathbb{H}$ and each $(\xi^1, \xi^2) \in \mathbb{R}^2$ with $\xi^1 < \xi^2$, $\mathcal{G}(z, \xi^1, \xi^2)$ can be extended to a continuous function of $\alpha \in (1, \infty)$.*

Proof. See Section 9. □

The CFT and screening considerations described in Section 4 suggest that \mathcal{G} is the Green's function for $\text{SLE}_\kappa(2)$ started from (ξ^1, ξ^2) ; that is, that $\mathcal{G}(z, \xi^1, \xi^2)$ provides the normalized probability that an $\text{SLE}_\kappa(2)$ path originating from ξ^1 with force point ξ^2 passes through an infinitesimal neighborhood of z . The following theorem establishes this rigorously.

Theorem 2.6 (Green's function for $SLE_\kappa(2)$). *Let $0 < \kappa \leq 4$. Let $-\infty < \xi^1 < \xi^2 < \infty$ and consider chordal $SLE_\kappa(2)$ started from (ξ^1, ξ^2) . Then, for each $z = x + iy \in \mathbb{H}$,*

$$\lim_{\epsilon \rightarrow 0} \epsilon^{d-2} \mathbf{P}_{\xi^1, \xi^2}^2 (\Upsilon_\infty(z) \leq \epsilon) = c_* \mathcal{G}(z, \xi^1, \xi^2), \quad (2.10)$$

where \mathbf{P}^2 is the $SLE_\kappa(2)$ measure, the function \mathcal{G} is defined in (2.8), and the constant $c_* = c_*(\kappa)$ is defined by

$$c_* = \frac{2}{\int_0^\pi \sin^{4a} x dx} = \frac{2\Gamma(1+2a)}{\sqrt{\pi}\Gamma(\frac{1}{2}+2a)} \quad \text{with} \quad a = \frac{2}{\kappa}. \quad (2.11)$$

The proof of Theorem 2.6 will be presented in Section 6.

Remark 2.7. The function $\mathcal{G}(z, \xi^1, \xi^2)$ can be written as

$$\mathcal{G}(z, \xi^1, \xi^2) = (\text{Im } z)^{d-2} h(\theta^1, \theta^2), \quad z \in \mathbb{H}, \quad -\infty < \xi^1 < \xi^2 < \infty, \quad (2.12)$$

where h is a function of $\theta^1 = \arg(z - \xi^1)$ and $\theta^2 = \arg(z - \xi^2)$. This is consistent with the expected translation invariance and scale covariance of the Green's function.

Remark 2.8. Formulas for $\mathcal{G}(z, \xi^1, \xi^2)$ when α is an integer are derived in Section 9. For $\alpha = 2, 3, 4$ (corresponding to $\kappa = 4, 8/3, 2$), the function $h(\theta^1, \theta^2)$ in (2.12) is given explicitly in (9.18), (9.19), and (9.20), respectively.

It is possible to derive an explicit expression for the Green's function for a system of two commuting SLEs as a consequence of Theorem 2.6. To this end, we need a correlation estimate which expresses the fact that it is very unlikely that both curves in a commuting system pass near a given point $z \in \mathbb{H}$.

Lemma 2.9. *Let $0 < \kappa \leq 4$. Then,*

$$\lim_{\epsilon \downarrow 0} \epsilon^{d-2} \mathbf{P}_{\xi^1, \xi^2}^2 (\Upsilon_\infty(z) \leq \epsilon) = \lim_{\epsilon \downarrow 0} \epsilon^{d-2} \left[\mathbf{P}_{\xi^1, \xi^2}^2 (\Upsilon_\infty(z) \leq \epsilon) + \mathbf{P}_{\xi^2, \xi^1}^2 (\Upsilon_\infty(z) \leq \epsilon) \right],$$

where $\mathbf{P}_{\xi^1, \xi^2}$ denotes the law of a system of two commuting SLE_κ in \mathbb{H} started from (ξ^1, ξ^2) and aiming for ∞ , and $\mathbf{P}_{\xi^1, \xi^2}^2$ denotes the law of chordal $SLE_\kappa(2)$ in \mathbb{H} started from (ξ^1, ξ^2) .

The proof of Lemma 2.9 will be given in Section 7.

Assuming Lemma 2.9, it follows immediately from Theorem 2.6 that the Green's function for a system of commuting SLEs started from $(-\xi, \xi)$ is given by

$$G_\xi(z) = \mathcal{G}(z, -\xi, \xi) + \mathcal{G}(-\bar{z}, -\xi, \xi), \quad z \in \mathbb{H}, \quad \xi > 0.$$

In other words, given a system of two commuting SLE_κ paths started from $-\xi$ and ξ respectively, $G_\xi(z)$ provides the normalized probability that at least one of the two curves passes through an infinitesimal neighborhood of z . We formulate this as a corollary.

Corollary 2.10 (Green's function for two commuting SLEs). *Let $0 < \kappa \leq 4$. Let $\xi > 0$ and consider a system of two commuting SLE_κ paths in \mathbb{H} started from $(-\xi, \xi)$ and growing towards ∞ . Then, for each $z = x + iy \in \mathbb{H}$,*

$$\lim_{\epsilon \rightarrow 0} \epsilon^{d-2} \mathbf{P}_{-\xi, \xi} (\Upsilon_\infty(z) \leq \epsilon) = c_* G_\xi(z), \quad (2.13)$$

where $d = 1 + \kappa/8$, the constant $c_* = c_*(\kappa)$ is given by (2.11), and the function G_ξ is defined for $z \in \mathbb{H}$ and $\xi > 0$ by

$$G_\xi(z) = \frac{1}{\hat{c}} y^{\alpha + \frac{1}{\alpha} - 2} |z + \xi|^{1-\alpha} |z - \xi|^{1-\alpha} \operatorname{Im} [e^{-i\pi\alpha} (I(z, -\xi, \xi) + I(-\bar{z}, -\xi, \xi))].$$

Remark 2.11. If the commuting system is started from two arbitrary points $(\xi^1, \xi^2) \in \mathbb{R}$ with $\xi^1 < \xi^2$, then it follows immediately from (2.13) and translation invariance that

$$\lim_{\epsilon \rightarrow 0} \epsilon^{d-2} \mathbf{P}_{\xi^1, \xi^2} (\Upsilon_\infty \leq \epsilon) = c_* G_{\frac{\xi^2 - \xi^1}{2}} \left(z - \frac{\xi^1 + \xi^2}{2} \right).$$

The proof of Theorem 2.6 will consist of establishing two independent propositions, which when combined imply Theorem 2.6. The first of these propositions (Proposition 2.12) establishes existence of a Green's function for $SLE_\kappa(\rho)$ and provides a representation for this Green's function in terms of an expectation with respect to two-sided radial SLE_κ . For the proof of Theorem 2.6, we only need this proposition for $\rho = 2$. However, since it is no more difficult to state and prove it for a suitable range of positive ρ , we consider the general case.

Proposition 2.12 (Existence and representation of Green's function for $SLE_\kappa(\rho)$). *Let $0 < \kappa \leq 4$ and $0 \leq \rho < 8 - \kappa$. Given two points $\xi^1, \xi^2 \in \mathbb{R}$ with $\xi^1 < \xi^2$, consider chordal $SLE_\kappa(\rho)$ started from (ξ^1, ξ^2) . Then, for each $z \in \mathbb{H}$,*

$$\lim_{\epsilon \downarrow 0} \epsilon^{d-2} \mathbf{P}_{\xi^1, \xi^2}^\rho (\Upsilon_\infty \leq \epsilon) = c_* G^\rho(z, \xi^1, \xi^2),$$

where the $SLE_\kappa(\rho)$ Green's function G^ρ is given by

$$G^\rho(z, \xi^1, \xi^2) = G(z - \xi^1) \mathbf{E}_{\xi^1, z}^* [M_T^{(\rho)}]. \quad (2.14)$$

Here $G(z) = (\operatorname{Im} z)^{d-2} \sin^{4a-1}(\arg z)$ is the Green's function for chordal SLE_κ in \mathbb{H} from 0 to ∞ , the martingale $M_t^{(\rho)}$ is defined in (3.7), $\mathbf{E}_{\xi^1, z}^*$ denotes expectation with respect to two-sided radial SLE_κ from ξ^1 through z , stopped at T , the hitting time of z , and the constant c_* is given by (2.11).

The next result (Proposition 2.13) shows that the function $\mathcal{G}(z, \xi^1, \xi^2)$ predicted by CFT and defined in (2.8) can be represented in terms of an expectation with respect to two-sided radial SLE_κ . Since this representation coincides with the representation in (2.14), Theorem 2.6 will follow immediately once we establish Propositions 2.12 and 2.13.

Proposition 2.13 (Representation of \mathcal{G}). *Let $0 < \kappa \leq 4$ and let $\xi^1, \xi^2 \in \mathbb{R}$ with $\xi^1 < \xi^2$. The function $\mathcal{G}(z, \xi^1, \xi^2)$ defined in (2.8) satisfies*

$$\mathcal{G}(z, \xi^1, \xi^2) = G(z - \xi^1) \mathbf{E}_{\xi^1, z}^* \left[M_T^{(2)} \right], \quad z \in \mathbb{H}, \quad 0 < \xi < \infty, \quad (2.15)$$

where $G(z) = (\operatorname{Im} z)^{d-2} \sin^{4a-1}(\arg z)$ is the Green's function for chordal SLE_κ in \mathbb{H} from 0 to ∞ and $\mathbf{E}_{\xi^1, z}^*$ denotes expectation with respect to two-sided radial SLE_κ from ξ^1 through z , stopped at T , the hitting time of z .

Remark 2.14. Note that equation (2.15) gives a formula for the two-sided radial SLE observable,

$$\mathbf{E}_{\xi^1, z}^* \left[M_T^{(2)} \right] = \frac{\mathcal{G}(z, \xi^1, \xi^2)}{G(z - \xi^1)}$$

and as a consequence we obtain smoothness and the fact that it satisfies the expected PDE.

The proofs of Propositions 2.12 and 2.13 are presented in Sections 6.1 and 6.2, respectively.

In Section 8.2, we obtain fusion formulas by letting $\xi \rightarrow 0+$. The formulas simplify for some values of κ .

Proposition 2.15 (Fused Green's functions). *Suppose $\kappa = 4, 8/3$, or 2 . Consider a system of two fused commuting SLE_κ paths in \mathbb{H} started from 0 and growing toward ∞ . Then, for each $z = x + iy \in \mathbb{H}$,*

$$\lim_{\epsilon \rightarrow 0} \epsilon^{d-2} \mathbf{P}_{0,0+}(\Upsilon_\infty(z) \leq \epsilon) = c_*(\mathcal{G}_f(z) + \mathcal{G}_f(-\bar{z})),$$

where $d = 1 + \kappa/8$, the constant $c_* = c_*(\kappa)$ is given by (2.11), and the function \mathcal{G}_f is defined by

$$\mathcal{G}_f(x + iy) = y^{d-2} h_f(\theta)$$

with $h_f(\theta)$ given explicitly by

$$h_f(\theta) = \begin{cases} \frac{2}{\pi}(\sin \theta - \theta \cos \theta) \sin \theta, & \kappa = 4, \\ \frac{8}{15\pi}(4\theta - 3 \sin 2\theta + 2\theta \cos 2\theta) \sin^2 \theta, & \kappa = 8/3, \\ \frac{1}{12\pi}(27 \sin \theta + 11 \sin 3\theta - 6\theta(9 \cos \theta + \cos 3\theta)) \sin^3 \theta, & \kappa = 2, \end{cases} \quad 0 < \theta < \pi.$$

2.3 Remarks

We end this section by making a few remarks.

- We believe the method used in this paper will generalize to produce analogous results for observables for $N \geq 3$ commuting SLE paths depending on one interior point. This would require $N - 1$ screening insertions, and the integrals will then be $N - 1$ iterated contour integrals.

- In [25, 26] screening integrals for SLE *boundary* observables (such as the ordered multipoint boundary Green’s function) are given and shown to be closely related to a particular *quantum group*. In fact, this algebraic structure is used to systematically make the difficult choices of integration contours. It seems reasonable to expect that a similar connection exists in our setting as well, allowing for an efficient generalization to several commuting SLE curves, but we will not pursue this here.
- Another way of viewing the system of two commuting SLEs growing towards ∞ is as one SLE path conditioned to hit a boundary point, also known as two-sided chordal SLE. Indeed, the extra $\rho = 2$ at the second seed point forces a $\rho = \kappa - 8$ at ∞ .
- A comparison with the representation of Appell’s F_1 function as an Euler integral (see e.g. [17]) shows that our screening integrals in the simplest case can be viewed as analytic continuations of Appell’s function. It is possible that our asymptotic estimates can be derived from known facts about these functions, but we have not been able to find the needed bounds in the literature. Besides, our approach generalizes to accomodate for multiple screening insertions, corresponding, for example, to several commuting SLEs. We refer to Section 10 for additional discussion.
- Suppose one has an SLE_κ martingale and wants to construct a similar martingale for $\text{SLE}_\kappa(\rho)$. The first idea that comes to mind is to try to “compensate” the SLE_κ martingale by multiplying by a differentiable process. In the cases we consider this does not give the correct observables, but rather corresponds to a change of coordinates moving the target point at ∞ .

3 Preliminaries

Unless specified otherwise, all complex powers are defined using the principal branch of the logarithm, that is, $z^\alpha = e^{\alpha(\ln|z| + i\text{Arg} z)}$ where $\text{Arg} z \in (-\pi, \pi]$. We write $z = x + iy$ and let

$$\partial = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \bar{\partial} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

We let $\mathbb{H} = \{z \in \mathbb{C} : \text{Im} z > 0\}$ and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open upper half-plane and the open unit disk, respectively. The open disk of radius $\epsilon > 0$ centered at $z \in \mathbb{C}$ will be denoted by $B_\epsilon(z) = \{w \in \mathbb{C} : |w - z| < \epsilon\}$. Throughout the paper, $c > 0$ and $C > 0$ will denote generic constants which may change within a computation.

Let D be a simply connected domain with two distinct boundary points p, q (prime ends). There is a conformal transformation $f : D \rightarrow \mathbb{H}$ taking p to 0 and q to ∞ ; in fact, f is determined only up to a final scaling. We choose one such f , but the quantities we define do not depend on the choice. Given $z \in D$, we define the conformal radius $r_D(z)$ of D seen from z by letting

$$\Upsilon_D(z) = \frac{\text{Im} f(z)}{|f'(z)|}, \quad r_D(z) = 2\Upsilon_D(z).$$

Schwarz’ lemma and Koebe’s 1/4 theorem give the distortion estimates

$$\text{dist}(z, \partial D)/2 \leq \Upsilon_D(z) \leq 2 \text{dist}(z, \partial D). \quad (3.1)$$

We define

$$S_{D,p,q}(z) = \sin[\arg f(z)], \quad S(z) = S_{\mathbb{H},0,\infty}(z),$$

and note that this is a conformal invariant. Suppose D is a Jordan domain and that J_-, J_+ are the boundary arcs $f^{-1}(\mathbb{R}_-)$ and $f^{-1}(\mathbb{R}_+)$, respectively. Let $\omega_D(z, E)$ denote the harmonic measure of E in D from z . Then it is easy to see that

$$S_{D,p,q}(z) \asymp \min\{\omega_D(z, J_-), \omega_D(z, J_+)\}, \quad (3.2)$$

with the implicit constants universal. By conformal invariance an analogous statement holds for any simply connected domain different from \mathbb{C} . We will use this relation several times without explicitly saying so in order to estimate $S_{D,p,q}$. In many places we will estimate harmonic measure using *the Beurling estimate*, see for example [21] Theorem IV.6.2 (with $\theta = 2\pi$).

We will also make use of *excursion measure*. Suppose D is analytic with two disjoint boundary arcs A, B . We define the excursion measure in D between A and B by

$$\mathcal{E}_D(A, B) = \int_A \partial_n \omega(\zeta, B) |d\zeta| = \int_B \partial_n \omega(\zeta, A) |d\zeta|,$$

where ω is harmonic measure and ∂_n denotes normal derivative in the inward pointing direction. For example, one has

$$\pi \mathcal{E}_{\mathbb{H}}([-x, 0], [1, y]) = \ln y - \ln \frac{y+x}{1+x},$$

so that

$$\pi \mathcal{E}_{\mathbb{H}}((-\infty, 0], [1, 1+x]) = \ln(1+x) = x + O(x^2), \quad (3.3)$$

as $x \downarrow 0$. Excursion measure is clearly a conformal invariant, and consequently we can define it in rough domains by mapping to the half plane and computing there.

3.1 Schramm-Loewner evolution

Let $0 < \kappa < 8$. Throughout the paper we will use the following parameters:

$$a = \frac{2}{\kappa}, \quad r = r_\kappa(\rho) = \frac{\rho}{\kappa} = \frac{\rho a}{2}, \quad d = 1 + \frac{1}{4a}, \quad \beta = 4a - 1.$$

We will also sometimes write $\alpha = 4a$. The assumption $\kappa = 2/a < 8$ implies that $\alpha > 1$.

We will work with the κ -dependent Loewner equation

$$\partial_t g_t(z) = \frac{a}{g_t(z) - \xi_t}, \quad g_0(z) = z, \quad (3.4)$$

where ξ_t , $t \geq 0$, is the (continuous) Loewner driving term. The solution is a family of conformal maps $(g_t(z))$ called the Loewner chain of ξ_t . The SLE_κ Loewner chain is obtained by taking ξ_t to be a standard Brownian motion and $a = 2/\kappa$. The SLE_κ path is the continuous curve connecting 0 with ∞ in \mathbb{H} defined by

$$\gamma(t) = \lim_{y \downarrow 0} g_t^{-1}(\xi_t + iy), \quad \gamma_t := \gamma[0, t].$$

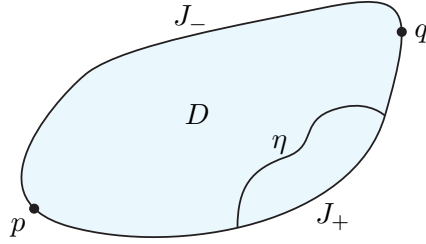


Figure 3. The domain D and the crosscut η of Lemma 3.2.

We write H_t for the simply connected domain given by taking the unbounded component of $\mathbb{H} \setminus \gamma_t$. Given a simply connected domain D with distinct boundary points p, q , we define chordal SLE_κ in D from p to q by conformal invariance. We write

$$S_t(z) = S_{H_t, \gamma(t), \infty}(z), \quad \Upsilon_t(z) = \Upsilon_{H_t}(z) = \frac{\text{Im } g_t(z)}{|g'_t(z)|}.$$

We will make use of the following sharp one-point estimate which also defines the Green's function for chordal SLE_κ , see [30] for a proof.

Lemma 3.1 (Green's function for chordal SLE_κ). *Suppose $0 < \kappa < 8$. Let γ be SLE_κ in D from p to q , where D is a simply connected domain with distinct boundary points (prime ends) p, q . Then there exists a constant $c > 0$ such that as $\epsilon \rightarrow 0$ the following estimate holds uniformly with respect to all $z \in D$ satisfying $\text{dist}(z, \partial D) \geq 2\epsilon$:*

$$\mathbf{P}(\Upsilon_\infty(z) \leq \epsilon) = c_* \epsilon^{2-d} G_D(z; p, q) [1 + O(\epsilon^c)],$$

where, by definition,

$$G_D(z; p, q) = \Upsilon_D(z)^{d-2} S_{D,p,q}^\beta(z)$$

is the Green's function for SLE_κ from p to q in D , and c_* is the constant defined in (2.11).

We also need to use a boundary estimate for SLE which is convenient to express in terms of excursion measure, see Lemma 4.5 of [30].

Lemma 3.2. *Let $0 < \kappa < 8$. Suppose D is a simply connected Jordan domain and let $p, q \in \partial D$ be two distinct boundary points. Write J_-, J_+ for the boundary arcs connecting q with p and p with q in the counterclockwise direction, respectively. Suppose η is a crosscut of D starting and ending on J_+ , see Figure 3. Then, if γ is chordal SLE_κ in D from p to q ,*

$$\mathbf{P}(\gamma_\infty \cap \eta \neq \emptyset) \leq c \mathcal{E}_{D \setminus \eta}(J_-, \eta)^\beta, \quad (3.5)$$

where $\beta = 4a - 1$ and the constant $c \in (0, \infty)$ is independent of D, p, q , and η .

3.1.1 $\text{SLE}_\kappa(\rho)$

Let $0 < \kappa < 8$ and $\rho \in \mathbb{R}$. We will work with $\text{SLE}_\kappa(\rho)$ as defined by weighting SLE_κ by a local martingale. Let $(\xi^1, \xi^2) \in \mathbb{R}^2$ be given with $\xi^1 < \xi^2$. Suppose ξ_t^1 is Brownian motion started from ξ^1 under the measure \mathbf{P} , with filtration \mathcal{F}_t . We refer to \mathbf{P} as the SLE_κ measure. Let (g_t) be the SLE_κ Loewner flow defined by equation (3.4) with $\xi_t = \xi_t^1$ and set

$$\xi_t^2 := g_t(\xi^2). \quad (3.6)$$

We call ξ^2 the force point. Define

$$\lambda(r) = \frac{r}{2a}(r - \beta), \quad \zeta(r) = \lambda(-r) - r = \frac{r}{2a}(r + 2a - 1).$$

Note that $\zeta \geq 0$ whenever $0 < \kappa \leq 4$ and $r \geq 0$. Itô's formula shows that

$$M_t^{(\rho)} = \left(\frac{\xi_t^2 - \xi_t^1}{\xi^2 - \xi^1} \right)^r g'_t(\xi^2)^{\zeta(r)} \quad (3.7)$$

is a local \mathbf{P} -martingale for any $\rho \in \mathbb{R}$. In fact,

$$\frac{dM_t^{(\rho)}}{M_t^{(\rho)}} = \frac{-r}{\xi_t^2 - \xi_t^1} d\xi_t^1.$$

The $\text{SLE}_\kappa(\rho)$ measure $\mathbf{P}^\rho = \mathbf{P}_{\xi^1, \xi^2}^\rho$ is defined by weighting \mathbf{P} by the martingale $M^{(\rho)}$, that is,

$$\mathbf{P}^\rho(V) = \mathbf{E}[M_t^{(\rho)} 1_V] \quad \text{for } V \in \mathcal{F}_t. \quad (3.8)$$

Then, using Girsanov's theorem, the equation for ξ_t^1 changes to

$$d\xi_t^1 = \frac{r}{\xi_t^1 - \xi_t^2} dt + dW_t, \quad (3.9)$$

where W_t is \mathbf{P}^ρ -Brownian motion. This is the defining equation for the driving term of $\text{SLE}_\kappa(\rho)$. (Since $M^{(\rho)}$ is a local martingale we need to stop the process before $M^{(\rho)}$ blows up; we will not always be explicit about this. We will not need to consider $\text{SLE}_\kappa(\rho)$ after the time the path hits the force point.) We refer to the Loewner chain driven by ξ_t^1 under \mathbf{P}^ρ as $\text{SLE}_\kappa(\rho)$ started from (ξ^1, ξ^2) . If ρ is sufficiently negative, the $\text{SLE}_\kappa(\rho)$ path will almost surely hit the force point. In this case it can be useful to reparametrize so that the quantity

$$C_t = C_t(\xi^2) = \frac{\xi_t^2 - O_t}{g'_t(\xi^2)}, \quad (3.10)$$

decays deterministically; this is called the radial parametrization in this context. Here O_t is defined as the image under g_t of the rightmost point in the hull at time t ; in particular, $O_t = g_t(0+)$ if $0 < \kappa \leq 4$, see, e.g., [1]. Geometrically C_t equals (1/4 times) the conformal

radius seen from ξ^2 in H_t after Schwarz reflection. We define a time-change $s(t)$ so that $\hat{C}_t := C_{s(t)} = e^{-at}$. A computation shows that if

$$A_t = \frac{\xi_t^2 - O_t}{\xi_t^2 - \xi_t^1}$$

then $s'(t) = (\hat{\xi}_t^2 - \hat{\xi}_t^1)^2 (\hat{A}_t^{-1} - 1)$, where $\hat{A}_t = A_{s(t)}$, see e.g. Section 2.2 of [1]. An important fact is that \hat{A}_t is positive recurrent with respect to $\text{SLE}_\kappa(\rho)$ if ρ is chosen appropriately.

Lemma 3.3. *Suppose $0 < \kappa < 8$ and $\rho < \kappa/2 - 4$. Consider $\text{SLE}_\kappa(\rho)$ started from $(0, 1)$. Then \hat{A}_t is positive recurrent with invariant density*

$$\pi_A(x) = c' x^{-\beta-a\rho} (1-x)^{2a-1}, \quad c' = \frac{\Gamma(2-2a-a\rho)}{\Gamma(2a)\Gamma(2-4a-a\rho)}.$$

In fact, there is $\alpha > 0$ such that if f is integrable with respect to the density π_A , then as $t \rightarrow \infty$,

$$\mathbf{E} [f(\hat{A}_t)] = c' \int_0^1 f(x) \pi_A(x) dx \left(1 + O(e^{-\alpha t})\right).$$

Proof. See Section 2 of [1] and Section 5.2 of [28]. □

3.1.2 Relationship between multiple SLE and $\text{SLE}_\kappa(\rho)$

Consider a system of two commuting chordal SLEs curves started from (ξ^1, ξ^2) ; recall (2.1) and (2.2). Suppose we first grow γ_2 up to capacity t . The conditional law of $g_t \circ \gamma_1$ is then an $\text{SLE}_\kappa(2)$ in \mathbb{H} started from (ξ_t^1, ξ_t^2) . In particular, the marginal law of γ_1 is that of an $\text{SLE}_\kappa(2)$ started from ξ^1 with force point ξ^2 . Indeed, if we choose the particular growth speeds $\lambda_1(t) \equiv a$ and $\lambda_2(t) \equiv 0$, then the defining equations (2.1) and (2.2) for commuting SLE reduce to

$$\begin{cases} \partial_t g_t(z) = \frac{a}{g_t(z) - \xi_t^1}, & g_0(z) = z, \\ d\xi_t^1 = \frac{a}{\xi_t^1 - \xi_t^2} dt + dB_t^1, & \xi_0^1 = \xi^1, \\ d\xi_t^2 = \frac{a}{\xi_t^2 - \xi_t^1} dt, & \xi_0^2 = \xi^2, \end{cases} \quad (3.11)$$

where B_t^1 is \mathbf{P} -Brownian motion. Evaluating the equation for $g_t(z)$ at $z = \xi^2$ we infer that $\xi_t^2 = g_t(\xi^2)$. Comparing this with the equations (3.6) and (3.9) defining $\text{SLE}_\kappa(\rho)$, we conclude that $\gamma_1(t)$ has the same distribution under the commuting SLE_κ measure \mathbf{P} as it has under the $\text{SLE}_\kappa(2)$ -measure \mathbf{P}^2 started from (ξ^1, ξ^2) .

3.1.3 Two-sided radial SLE and radial parametrization

Recall that if $z \in \mathbb{H}$ is fixed then the SLE_κ Green's function in H_t equals

$$G_t = G_t(z) = \Upsilon_t^{d-2}(z) S_t^\beta(z), \quad (3.12)$$

which is a covariant \mathbf{P} -martingale. *Two-sided radial SLE* in \mathbb{H} through z is the process obtained by weighting chordal SLE_κ by G_t . (This is the same as $\text{SLE}_\kappa(\kappa - 8)$ with force

point $z \in \mathbb{H}$.) Since two-sided radial SLE approaches its target point, it is natural to use the *radial parametrization*, so that the conformal radius (seen from z) decays deterministically. More precisely, we change time so that $\Upsilon_{s(t)}(z) = e^{-2at}$; this parametrization depends on z . The Loewner equation implies

$$d \ln \Upsilon_t = -2a \frac{y_t^2}{|z_t|^4} dt, \quad z_t = x_t + iy_t = g_t(z) - \xi_t^1.$$

Hence $s'(t) = |\tilde{z}_t|^4 / \tilde{y}_t^2$, where $\tilde{S}_t = S_{s(t)}$, $\tilde{z}_t = z_{s(t)}$, etc., denote the time-changed processes. Using that

$$d\Theta_t = (1 - 2a) \frac{x_t y_t}{|z_t|^4} dt + \frac{y_t}{|z_t|^2} d\xi_t^1,$$

we find that $\tilde{\Theta}_t = \Theta_{s(t)}$ satisfies

$$d\tilde{\Theta}_t = (1 - 2a) \cot \tilde{\Theta}_t dt + d\tilde{W}_t,$$

where \tilde{W}_t is standard \mathbf{P} -Brownian motion. The time-changed martingale can be written

$$\tilde{G}_t = e^{-2a(d-2)t} \tilde{S}_t^\beta. \quad (3.13)$$

The *two-sided radial SLE $_\kappa$ measure* $\mathbf{P}^* = \mathbf{P}_z^*$ is defined by weighting chordal SLE $_\kappa$ by \tilde{G}_t , that is,

$$\mathbf{P}^*(V) = \tilde{G}_0^{-1} \mathbf{E}[\tilde{G}_t 1_V], \quad V \in \tilde{\mathcal{F}}_t. \quad (3.14)$$

This produces two-sided radial SLE $_\kappa$ in the radial parametrization.

Since $d\tilde{G}_t = \beta \tilde{G}_t \cot(\tilde{\Theta}_t) d\tilde{W}_t$, Girsanov's theorem implies that the equation for $\tilde{\Theta}_t$ changes to the *radial Bessel equation* under the new measure \mathbf{P}^* :

$$d\tilde{\Theta}_t = 2a \cot \tilde{\Theta}_t dt + d\tilde{B}_t,$$

where \tilde{B}_t is \mathbf{P}^* -standard Brownian motion.

We will use the following lemma about the radial Bessel equation, see, e.g., Section 3 of [29].

Lemma 3.4. *Let $0 < \kappa < 8$, $a = 2/\kappa$ and suppose the process (Θ_t) is a solution to the SDE*

$$d\Theta_t = 2a \cot \Theta_t dt + dB_t, \quad \Theta_0 = \Theta. \quad (3.15)$$

Then Θ_t is positive recurrent with invariant density

$$\psi(x) = \frac{c_*}{2} \sin^{4a} x,$$

where c_ is the constant in (2.11). In fact, there is $\alpha > 0$ such that if f is integrable with respect to the density ψ , then as $t \rightarrow \infty$,*

$$\mathbf{E}[f(\Theta_t)] = \int_0^\pi f(x) \psi(x) dx (1 + O(e^{-\alpha t})),$$

where the error term does not depend on Θ_0 .

4 Martingale observables as CFT correlation functions

4.1 Screening

The CFT framework of Kang and Makarov [23] can be used to generate martingale observables for SLE systems, see in particular Lecture 14 of [23]. The framework of [23] has been extended to incorporate several commuting SLEs started from different points in [3]. We will use the screening method [13] which produces observables in the form of contour integrals, which we call screening integrals or Coulomb gas integrals. Let us briefly and informally describe the method. From the CFT perspective (we work in the framework of [23]), one starts from a CFT correlation function with appropriate field insertions giving a corresponding (known) SLE_κ martingale. Adding additional paths means inserting additional boundary fields. This will create observables for the system of SLEs. But in the cases we consider the extra fields change the boundary behavior so that the new observable does not encode the desired geometric information anymore. To remedy this, carefully chosen auxiliary fields are inserted and then integrated out along integration contours. The correct choices of insertions and integration contours depend on the particular problem, and different choices correspond to solutions with different boundary behavior.

Remark 4.1. We mention in passing that from a different point of view, it is known that the Gaussian free field with suitable boundary data can be coupled with SLE paths as “local sets” for the field [31]. By the nature of the coupling, correlation functions for the field should give rise to SLE martingales. The appropriate boundary conditions can be understood as insertions of suitable fields on the boundary.

In what follows, we briefly summarize how we used these ideas to arrive at the explicit expressions (2.5) and (2.8) for the Schramm probability $P(z, \xi)$ and the Green’s function $\mathcal{G}(z, \xi^1, \xi^2)$, respectively. Since the discussion is purely motivational, except for Lemma 4.2 we make no attempt in this section to be complete or rigorous (this is in contrast to the other sections of the paper which are rigorous). We refer to [23, 3] for an introduction to the underlying CFT framework and we will use notation from this reference.

Consider a system of two commuting SLEs started from $(\xi^1, \xi^2) \in \mathbb{R}^2$. If $\lambda_1(t)$ and $\lambda_2(t)$ denote the growth speeds of the two curves, the evolution of the system is described by equations (2.1) and (2.2). In the conformal field theory language of [23], the presence of two commuting SLE curves in \mathbb{H} started from ξ^1 and ξ^2 corresponds to the insertion of the operator

$$\mathcal{O}(\xi^1, \xi^2) = V_{\star, (b)}^{i\sqrt{a}}(\xi^1) V_{\star, (b)}^{i\sqrt{a}}(\xi^2),$$

where $V_{\star, (b)}^{i\sigma}(z)$ denotes a rooted vertex field inserted at z (see [23], p. 96) and the parameter b satisfies the relation

$$2\sqrt{a}(\sqrt{a} + b) = 1, \quad a = 2/\kappa. \quad (4.1)$$

Notice that we define $a = 2/\kappa$ while [23] defines “ a ” by $\sqrt{2/\kappa}$. The framework of [23] (or rather an extension of this framework to the case of multiple curves [3]) suggests that if $\{z_j\}_1^n \subset \mathbb{C}$ are points and $\{X_j\}_1^n$ are fields satisfying certain properties, then the correlation function

$$M_t^{(z_1, \dots, z_n)} = \hat{\mathbf{E}}_{\mathcal{O}(\xi_t^1, \xi_t^2)}^{\mathbb{H}}[(X_1 || g_t^{-1})(z_1) \cdots (X_n || g_t^{-1})(z_n)] \quad (4.2)$$

is a (local) martingale observable for the system when evaluated in the “Loewner charts” (g_t) . It turns out that the observables relevant for Schramm’s formula and for the Green’s function belong to a class of correlation functions of the form

$$M_t^{(z,u)} = \hat{\mathbf{E}}_{\mathcal{O}(\xi_t)}[(V_{\star,(b)}^{i\sigma_1}||g_t^{-1})(z)\overline{(V_{\star,(b)}^{i\sigma_2}||g_t^{-1})(z)}(V_{\star,(b)}^{is}||g_t^{-1})(u)], \quad (4.3)$$

where $z \in \mathbb{H}$, $u \in \mathbb{C}$, and $\sigma_1, \sigma_2, s \in \mathbb{R}$ are real constants. We will later integrate out the variable u , but it is essential to include the screening field $(V_{\star,(b)}^{is}||g_t^{-1})(u)$ in the definition (4.3) in order to arrive at observables with the appropriate conformal dimensions at z and at infinity. The observable $M_t^{(z,u)}$ can be written as

$$M_t^{(z,u)} = g'_t(z)^{\frac{\sigma_1^2}{2}-\sigma_1 b} \overline{g'_t(z)^{\frac{\sigma_2^2}{2}-\sigma_2 b}} g'_t(u)^{\frac{s^2}{2}-sb} A(Z_t, \xi_t^1, \xi_t^2, U_t), \quad (4.4)$$

where $Z_t = g_t(z)$, $U_t = g_t(u)$, and the function $A(z, \xi^1, \xi^2, u)$ is defined by

$$\begin{aligned} A(z, \xi^1, \xi^2, u) &= (z - \bar{z})^{\sigma_1 \sigma_2} [(z - \xi^1)(z - \xi^2)]^{\sigma_1 \sqrt{a}} [(\bar{z} - \xi^1)(\bar{z} - \xi^2)]^{\sigma_2 \sqrt{a}} \\ &\quad \times (z - u)^{\sigma_1 s} (\bar{z} - u)^{\sigma_2 s} [(u - \xi^1)(u - \xi^2)]^{s \sqrt{a}}. \end{aligned} \quad (4.5)$$

The following lemma confirms that the CFT generated observable $M_t^{(z,u)}$ is indeed a local martingale for any choice of $z, u \in \mathbb{H}$ and $\sigma_1, \sigma_2, s \in \mathbb{R}$.

Lemma 4.2. *Let $z, u \in \mathbb{H}$ be two distinct points in the upper half-plane and let $\sigma_1, \sigma_2, s \in \mathbb{R}$ be real numbers. For any choice of the growth speeds $\lambda_j(t)$, $j = 1, 2$, the function $M_t^{(z,u)}$ defined in (4.4) is a local martingale for the system of commuting SLEs started from (ξ^1, ξ^2) .*

Proof. It is enough to show that the drift coefficient D_t in the expression

$$dM_t^{(z,u)} = D_t dt + E_t^1 dB_t^1 + E_t^2 dB_t^2$$

vanishes. Writing $M := M_t^{(z,u)}$, equations (2.1) and (2.2) together with Itô’s formula imply

$$\begin{aligned} D_t &= \frac{\partial M}{\partial Z_t} \frac{dZ_t}{dt} + \frac{\partial M}{\partial \bar{Z}_t} \frac{d\bar{Z}_t}{dt} + \frac{\partial M}{\partial U_t} \frac{dU_t}{dt} + \frac{\partial M}{\partial g'_t(z)} \frac{\partial g'_t(z)}{\partial t} + \frac{\partial M}{\partial \overline{g'_t(z)}} \frac{\partial \overline{g'_t(z)}}{\partial t} + \frac{\partial M}{\partial g'_t(u)} \frac{\partial g'_t(u)}{\partial t} \\ &\quad + \sum_{j=1}^2 \frac{\partial M}{\partial \xi_t^j} \frac{\partial \xi_t^j}{\partial t} + \frac{\kappa}{4} \sum_{j=1}^2 \lambda_j(t) \frac{\partial^2 M}{\partial^2 \xi_t^j}, \end{aligned} \quad (4.6)$$

where $Z_t = X_t + iY_t$ and

$$\begin{aligned} \frac{dZ_t}{dt} &= \sum_{j=1}^2 \frac{\lambda_j(t)}{Z_t - \xi_t^j}, & \frac{d\bar{Z}_t}{dt} &= \sum_{j=1}^2 \frac{\lambda_j(t)}{\bar{Z}_t - \xi_t^j}, & \frac{dU_t}{dt} &= \sum_{j=1}^2 \frac{\lambda_j(t)}{U_t - \xi_t^j}, \\ \frac{\partial g'_t(z)}{\partial t} &= - \sum_{j=1}^2 \frac{g'_t(z) \lambda_j(t)}{(Z_t - \xi_t^j)^2}, & \frac{\partial \overline{g'_t(z)}}{\partial t} &= - \sum_{j=1}^2 \frac{\overline{g'_t(z)} \lambda_j(t)}{(\bar{Z}_t - \xi_t^j)^2}, \end{aligned}$$

$$\frac{\partial g'_t(u)}{\partial t} = - \sum_{j=1}^2 \frac{g'_t(u) \lambda_j(t)}{(U_t - \xi_t^j)^2}, \quad \frac{\partial \xi_t^1}{\partial t} = \frac{\lambda_1(t) + \lambda_2(t)}{\xi_t^1 - \xi_t^2}, \quad \frac{\partial \xi_t^2}{\partial t} = \frac{\lambda_1(t) + \lambda_2(t)}{\xi_t^2 - \xi_t^1}. \quad (4.7)$$

Substituting the expressions in (4.7) into (4.6), a direct computation shows that $D_t = 0$. \square

Since (4.4) is a local martingale for each value of the screening variable u , we expect the integrated observable

$$\mathcal{M}_t^{(z)} = \int_{\gamma} M_t^{(z,u)} du \quad (4.8)$$

to be a local martingale for any choice of $z \in \mathbb{H}$, $\sigma_1, \sigma_2, s \in \mathbb{R}$, and of the integration contour γ , at least as long as the integral in (4.8) converges and the branches of the complex powers in (4.5) are consistently defined. The integral in (4.8) is referred to as a “screening” integral.

By choosing $\lambda_2(t) = 0$ in Lemma 4.2, we see that the observable $\mathcal{M}_t^{(z)}$ defined in (4.8) is a local martingale for $\text{SLE}_{\kappa}(2)$ started from (ξ^1, ξ^2) . We next describe how the martingales relevant for Schramm’s formula and for the Green’s function for $\text{SLE}_{\kappa}(2)$ arise as special cases of $\mathcal{M}_t^{(z)}$ corresponding to particular choices of $\sigma_1, \sigma_2, s \in \mathbb{R}$ and of the contour γ .

4.2 Prediction of Schramm’s formula

In order to obtain the local martingale relevant for Schramm’s formula we choose the following values for the parameters (“charges”) in (4.4):

$$\sigma_1 = -2\sqrt{a}, \quad \sigma_2 = 2b, \quad s = -2\sqrt{a}. \quad (4.9)$$

The choice (4.9) can be motivated as follows. First of all, by choosing $s = -2\sqrt{a}$ we ensure that $s^2/2 - sb = 1$ (see (4.1)). This implies that $\mathcal{M}_t^{(z)}$ involves the one-form $g'_t(u)^{s^2/2 - sb} du = g'_t(u) du$. After integration with respect to du this leads to a conformally invariant screening integral. To motivate the choices of σ_1 and σ_2 , let $P(z, \xi^1, \xi^2)$ denote the probability that the point $z \in \mathbb{H}$ lies to the left of an $\text{SLE}_{\kappa}(2)$ -path started from (ξ^1, ξ^2) . Then we expect $\partial_z P$ to be a martingale observable with conformal dimensions

$$\lambda(z) = 1, \quad \lambda_*(z) = 0, \quad \lambda_{\infty} = 0. \quad (4.10)$$

The parameters in (4.9) are chosen so that the observable $\mathcal{M}_t^{(z)}$ in (4.8) has the conformal dimensions in (4.10). We emphasize that it is the inclusion of the screening field in (4.3) that makes it possible to obtain these dimensions. In particular, by including it we can have $\lambda_{\infty} = 0$. We have considered the derivative $\partial_z P$ instead of P because then we are able to construct a nontrivial martingale with the correct dimensions.

In the special case when the parameters σ_1, σ_2, s are given by (4.9), the local martingale (4.8) takes the form

$$\mathcal{M}_t^{(z)} = g'_t(z) (Z_t - \bar{Z}_t)^{\alpha-2} (Z_t - \xi_t^1)^{-\frac{\alpha}{2}} (Z_t - \xi_t^2)^{-\frac{\alpha}{2}} (\bar{Z}_t - \xi_t^1)^{1-\frac{\alpha}{2}} (\bar{Z}_t - \xi_t^2)^{1-\frac{\alpha}{2}}$$

$$\times \int_{\gamma} (Z_t - U_t)^{\alpha} (\bar{Z}_t - U_t)^{\alpha-2} [(U_t - \xi_t^1)(U_t - \xi_t^2)]^{-\frac{\alpha}{2}} g'_t(u) du. \quad (4.11)$$

We expect from the above discussion that there exists an appropriate choice of the integration contour γ in (4.8) such that $\partial_z P(z, \xi^1, \xi^2) = \text{const} \times \mathcal{M}_0^{(z)}$, that is, we expect

$$\begin{aligned} \partial_z P(z, \xi^1, \xi^2) &= c(\kappa) y^{\alpha-2} (z - \xi^1)^{-\frac{\alpha}{2}} (z - \xi^2)^{-\frac{\alpha}{2}} (\bar{z} - \xi^1)^{1-\frac{\alpha}{2}} (\bar{z} - \xi^2)^{1-\frac{\alpha}{2}} \\ &\times \int_{\gamma} (u - z)^{\alpha} (u - \bar{z})^{\alpha-2} (u - \xi^1)^{-\frac{\alpha}{2}} (u - \xi^2)^{-\frac{\alpha}{2}} du, \end{aligned}$$

where $c(\kappa)$ is a complex constant. Setting $\xi^1 = 0$ and $\xi^2 = \xi$ in this formula, we arrive at the prediction (2.5) for the Schramm probability $P(z, \xi)$. Indeed, the integration with respect to x in (2.5) recovers P from $\partial_z P$ and ensures that P tends to zero as $\text{Re } z \rightarrow \infty$. On the other hand, the choice of the integration contour from \bar{z} to z in (2.4) is mandated by the requirement that $P(z, \xi)$ should satisfy the correct boundary conditions as z approaches the real axis. Finally, $P(z, \xi)$ must be a real-valued function tending to 1 as $\text{Re } z \rightarrow -\infty$; this fixes the constant $c(\kappa)$.

4.3 Prediction of the Green's function

In order to obtain the local martingale relevant for the $\text{SLE}_{\kappa}(2)$ Green's function, we choose the following values for the parameters in (4.4):

$$\sigma_1 = b - \sqrt{a}, \quad \sigma_2 = b - \sqrt{a}, \quad s = -2\sqrt{a}, \quad (4.12)$$

As in the case of Schramm's formula, the choice $s = -2\sqrt{a}$ ensures that $\mathcal{M}_t^{(z)}$ involves the one-form $g'_t(u)du$. Moreover, if we let $\mathcal{G}(z, \xi^1, \xi^2)$ denote the Green's function for $\text{SLE}_{\kappa}(2)$ started from (ξ^1, ξ^2) , then we expect \mathcal{G} to have the conformal dimensions (cf. page 124 in [23])

$$\lambda(z) = \lambda_*(z) = \frac{2-d}{2}, \quad \lambda_{\infty} = 0. \quad (4.13)$$

The parameters σ_1 and σ_2 in (4.12) are determined so that the observable $\mathcal{M}_t^{(z)}$ in (4.8) has the conformal dimensions in (4.13). For example, a generalization of Proposition 15.5 in [23] to the case of two curves implies that $\lambda_{\infty} = (2\sqrt{a} - b)\Sigma + \frac{\Sigma^2}{2} = 0$ where $\Sigma = \sigma_1 + \sigma_2 - 2\sqrt{a}$.

Remark 4.3. We can see here that the choice $\rho = 2$ is special: we have only two possible ways to add one screening field, corresponding to $s = -2\sqrt{a}$ or $s = 1/\sqrt{a}$. But the extra $\rho = 2$ corresponds to additional charges $\sigma = \sigma_* = 2/\sqrt{8\kappa}$ (we are using $\sigma = \rho/\sqrt{8\kappa}$, so at infinity we have an additional charge $\sigma + \sigma_* = 2\sqrt{a}$). We see that the $\rho = 2$ charge can be screened by only one screening field. We can also see that if we add more ρ 's, they can be screened if their charges sum up to $2\sqrt{a}$. Also this suggests that every SLE_{κ} observable with $\lambda_q = 0$ gives an $\text{SLE}_{\kappa}(2)$ observable with $\lambda_q = 0$ after screening. Similarly, since adding n additional $\rho_j = 2$ gives additional charges at ∞ of $2n\sqrt{a}$, one could expect that one can construct a martingale for a system of n SLEs by adding n screening charges.

In the special case when the parameters σ_1, σ_2, s are given by (4.12), the local martingale (4.8) takes the form

$$\mathcal{M}_t^{(z)} = |g'_t(z)|^{2-d} \int_{\gamma} \mathcal{A}(Z_t, \xi_t^1, \xi_t^2, g_t(u)) g'_t(u) du, \quad (4.14)$$

where

$$\begin{aligned} \mathcal{A}(z, \xi^1, \xi^2, u) &= (z - \bar{z})^{\alpha + \frac{1}{\alpha} - 2} |z - \xi^1|^{-\beta} |z - \xi^2|^{-\beta} \\ &\quad \times (z - u)^{\beta} (\bar{z} - u)^{\beta} [(u - \xi^1)(u - \xi^2)]^{-\frac{\alpha}{2}}. \end{aligned} \quad (4.15)$$

We expect from the above discussion that there exists an appropriate choice of the integration contour γ in (4.8) such that $\mathcal{G}(z, \xi^1, \xi^2) = \text{const} \times \mathcal{M}_0^{(z)}$, that is, we expect

$$\mathcal{G}(z, \xi^1, \xi^2) = c(\kappa) y^{\alpha + \frac{1}{\alpha} - 2} |z - \xi^1|^{-\beta} |z - \xi^2|^{-\beta} J(z, \xi^1, \xi^2), \quad (4.16)$$

where

$$J(z, \xi^1, \xi^2) = \int_{\gamma} (u - z)^{\beta} (u - \bar{z})^{\beta} (u - \xi^1)^{-\frac{\alpha}{2}} (\xi^2 - u)^{-\frac{\alpha}{2}} du$$

and $c(\kappa)$ is a complex constant. By requiring that G satisfy the correct boundary conditions, we arrive at the prediction (2.8) for the Green's function for $\text{SLE}_{\kappa}(2)$. The trickiest step is the determination of the appropriate screening contour γ . This contour must be chosen so that the Green's function satisfies the appropriate boundary conditions as (z, ξ^1, ξ^2) approach the boundary of the domain $\mathbb{H} \times \{-\infty < \xi^1 < \xi^2 < \infty\}$. The complete verification that the Pochhammer integration contour in (2.4) leads to the correct boundary behavior is presented in Lemma 6.2 and relies on a complicated analysis of integral asymptotics. We first arrived at the Pochhammer contour in (2.4) via the following simpler argument.

Let $\mathcal{G}_{\xi}(z) = \mathcal{G}(z, -\xi, \xi)$, $J_{\xi}(z) = J(z, -\xi, \xi)$. Let also $I_{\xi}(z) = I(z, -\xi, \xi)$ where I is the function defined in (2.7), i.e.,

$$I_{\xi}(z) = \int_A^{(z+, \xi+, z-, \xi-)} (u - z)^{\alpha-1} (u - \bar{z})^{\alpha-1} (\xi + u)^{-\frac{\alpha}{2}} (\xi - u)^{-\frac{\alpha}{2}} du. \quad (4.17)$$

We make the ansatz that

$$J_{\xi}(z) = \sum_{i=1}^4 c_i(\kappa) \int_{\gamma_i} (u - z)^{\alpha-1} (u - \bar{z})^{\alpha-1} (\xi + u)^{-\frac{\alpha}{2}} (\xi - u)^{-\frac{\alpha}{2}} du,$$

where the contours $\{\gamma_i\}_1^4$ are Pochhammer contours surrounding the pairs (ξ, z) , (ξ, \bar{z}) , $(-\xi, z)$, and $(-\xi, \bar{z})$, respectively. The integral involving the pair (ξ, z) is $I_{\xi}(z)$. The integrals involving the pairs $(\pm\xi, z)$ are related via complex conjugation to the integrals involving the pairs $(\pm\xi, \bar{z})$. Moreover, by performing the change of variables $u \rightarrow -\bar{u}$, we see that the integral involving the pair $(-\xi, z)$ can be expressed in terms of $I(-\bar{z})$. Thus, using the requirement that $J(z, \xi)$ be real-valued, we can without loss of generality assume that $J(z, \xi)$ is a real linear combination of the real and imaginary parts of $I_{\xi}(z)$ and $I_{\xi}(-\bar{z})$.

At this stage it is convenient, for simplicity, to assume $4 < \kappa < 8$ so that $1 < \alpha < 2$. Then we can collapse the contour in the definition (4.17) of $I_\xi(z)$ onto a curve from ξ from z ; this gives

$$I_\xi(z) = (1 - e^{2i\pi\alpha} + e^{i\pi\alpha} - e^{-i\pi\alpha})\hat{I}_\xi(z),$$

where $\hat{I}_\xi(z)$ is defined by

$$\hat{I}_\xi(z) = \int_\xi^z (u - z)^{\alpha-1} (u - \bar{z})^{\alpha-1} (\xi + u)^{-\frac{\alpha}{2}} (\xi - u)^{-\frac{\alpha}{2}} du.$$

Since \hat{I} obeys the symmetry $\text{Im } \hat{I}_\xi(z) = \text{Im } \hat{I}_\xi(-\bar{z})$, our ansatz takes the form

$$J_\xi(z) = A_1 \text{Re } \hat{I}_\xi(z) + A_2 \text{Re } \hat{I}_\xi(-\bar{z}) + A_3 \text{Im } \hat{I}_\xi(z), \quad (4.18)$$

where $A_j = A_j(\kappa)$, $j = 1, 2, 3$, are real constants.

Up to factors which are independent of y , we expect the Green's function $\mathcal{G}_\xi(z)$ to satisfy

$$\mathcal{G}_\xi(x + iy) \sim y^{d-2} = y^{\frac{1}{\alpha}-1}, \quad y \rightarrow \infty, \quad x \text{ fixed}, \quad (4.19a)$$

$$\mathcal{G}_\xi(\xi + iy) \sim y^{d-2} y^{\beta+2a} = y^{\frac{1}{\alpha} + \frac{3\alpha}{2} - 2}, \quad y \downarrow 0. \quad (4.19b)$$

Indeed, since the influence of the force point ξ^2 goes to zero as $\text{Im } \gamma(t)$ becomes large, the first relation follows by comparison with SLE_κ . The second relation can be motivated by noticing that the boundary exponent for $\text{SLE}_\kappa(\rho)$ at the force point ξ^2 is $\beta + \rho a$, see Lemma 7.3. In terms of $J_\xi(z)$, the estimates (4.19) translate into

$$J_\xi(x + iy) \sim y^{\alpha-1}, \quad y \rightarrow \infty, \quad x \text{ fixed}, \quad (4.20a)$$

$$J_\xi(\xi + iy) \sim y^{\frac{3\alpha}{2}-1}, \quad y \downarrow 0. \quad (4.20b)$$

We will use these conditions to fix the values of the A_j 's.

We obtain one constraint on the A_j 's by considering the asymptotics of $J_\xi(iy)$ as $y \rightarrow \infty$. Indeed, for $x = 0$ we have

$$\begin{aligned} \hat{I}_\xi(iy) &= \int_\xi^{iy} (u^2 + y^2)^{\alpha-1} (\xi^2 - u^2)^{-\frac{\alpha}{2}} du \\ &= \frac{i}{2} \sqrt{\pi} \xi^{-\alpha} y^{2\alpha-2} \left\{ \frac{y\Gamma(\alpha)}{\Gamma(\alpha + \frac{1}{2})} {}_2F_1\left(\frac{1}{2}, \frac{\alpha}{2}, \alpha + \frac{1}{2}, -\frac{y^2}{\xi^2}\right) \right. \\ &\quad \left. + \frac{i\xi\Gamma(1 - \frac{\alpha}{2})}{\Gamma(\frac{3}{2} - \frac{\alpha}{2})} {}_2F_1\left(\frac{1}{2}, 1 - \alpha, \frac{3}{2} - \frac{\alpha}{2}, -\frac{\xi^2}{y^2}\right) \right\}, \end{aligned}$$

where ${}_2F_1$ denotes the standard hypergeometric function. This implies

$$\begin{aligned} \hat{I}_\xi(iy) &= y^{\alpha-1} \left(\frac{i\Gamma(\frac{1}{2} - \frac{\alpha}{2})\Gamma(\alpha)}{2\Gamma(\frac{\alpha+1}{2})} + O\left(\frac{1}{y^2}\right) \right) \\ &\quad + y^{2(\alpha-1)} \left(-\frac{\pi^{3/2}\xi^{1-\alpha}(\csc(\frac{\pi\alpha}{2}) + i\sec(\frac{\pi\alpha}{2}))}{2(\Gamma(\frac{3}{2} - \frac{\alpha}{2})\Gamma(\frac{\alpha}{2}))} + O\left(\frac{1}{y^2}\right) \right), \quad y \rightarrow \infty. \end{aligned}$$

Substituting this expansion into (4.18), we find an expression for $J_\xi(iy)$ involving two terms which are proportional to $y^{2(\alpha-1)}$ and $y^{\alpha-1}$, respectively, as $y \rightarrow \infty$. In order to satisfy the condition (4.20a), we must choose the A_j so that the coefficient of the larger term involving $y^{2(\alpha-1)}$ vanishes. This leads to the relation

$$\frac{A_1 + A_2}{A_3} = -\tan \frac{\pi\alpha}{2}. \quad (4.21)$$

We obtain a second constraint on the A_j 's by considering the asymptotics of $J_\xi(iy)$ as $z \rightarrow \xi$. Indeed, for $x = \xi$ we have

$$\hat{I}_\xi(\xi + iy) = e^{\frac{i\pi}{2}(1+\frac{\alpha}{2})} \int_0^y (y^2 - s^2)^{\alpha-1} (2\xi + is)^{\alpha-1} s^{-\frac{\alpha}{2}} ds.$$

Hence

$$\begin{aligned} \hat{I}_\xi(\xi + iy) &\sim e^{\frac{i\pi}{2}(1+\frac{\alpha}{2})} (2\xi)^{-\frac{\alpha}{2}} \int_0^y (y^2 - s^2)^{\alpha-1} s^{-\frac{\alpha}{2}} ds \\ &= \frac{2^{-\frac{\alpha}{2}-1} e^{\frac{1}{4}i\pi(\alpha+2)} \xi^{-\alpha/2} \Gamma\left(\frac{1}{2} - \frac{\alpha}{4}\right) \Gamma(\alpha)}{\Gamma\left(\frac{3\alpha}{4} + \frac{1}{2}\right)} y^{\frac{3\alpha}{2}-1}, \quad y \downarrow 0, \quad \xi > 0. \end{aligned} \quad (4.22)$$

Similarly, for $x = -\xi$, we have

$$\begin{aligned} \hat{I}(-\xi + iy, \xi) &= \int_\xi^{-\xi} ((u + \xi)^2 + y^2)^{\alpha-1} (\xi + u)^{-\frac{\alpha}{2}} (\xi - u)^{-\frac{\alpha}{2}} du \\ &\quad + \int_0^y (i(s - y))^{\alpha-1} (i(s + y))^{\alpha-1} (is)^{-\frac{\alpha}{2}} (2\xi - is)^{-\frac{\alpha}{2}} i ds. \end{aligned}$$

Hence

$$\begin{aligned} \hat{I}(-\xi + iy, \xi) &\sim - \int_{-\xi}^\xi (u + \xi)^{\frac{3\alpha}{2}-2} (\xi - u)^{-\frac{\alpha}{2}} du + e^{\frac{i\pi}{2}(1-\frac{\alpha}{2})} (2\xi)^{-\frac{\alpha}{2}} \int_0^y (y^2 - s^2)^{\alpha-1} s^{-\frac{\alpha}{2}} ds \\ &= 2\xi^{\alpha-1} \left(\frac{{}_2F_1\left(1, 2 - \frac{3\alpha}{2}, 2 - \frac{\alpha}{2}; -1\right)}{\alpha - 2} + \frac{{}_2F_1\left(1, \frac{\alpha}{2}, \frac{3\alpha}{2}; -1\right)}{2 - 3\alpha} \right) \\ &\quad + \frac{i 2^{-\frac{\alpha}{2}-1} e^{-\frac{1}{4}i\pi\alpha} \xi^{-\alpha/2} \Gamma\left(\frac{1}{2} - \frac{\alpha}{4}\right) \Gamma(\alpha)}{\Gamma\left(\frac{3\alpha}{4} + \frac{1}{2}\right)} y^{\frac{3\alpha}{2}-1}, \quad y \downarrow 0, \quad \xi > 0. \end{aligned} \quad (4.23)$$

Substituting the expansions (4.22) and (4.23) into (4.18), we find an expression for $J_\xi(\xi + iy)$ involving two terms which are of order $O(y^{\frac{3\alpha}{2}-1})$ and $O(1)$, respectively, as $y \rightarrow 0$. In order to satisfy the condition (4.20b), we must choose the A_j so that the coefficient of the larger term of $O(1)$ vanishes. This implies

$$A_2 = 0. \quad (4.24)$$

Using the constraints (4.21) and (4.24), the expression (4.18) becomes

$$J_\xi(z) = B_1 \text{Im} \left(e^{-\frac{i\pi\alpha}{2}} \hat{I}_\xi(z) \right) = B_2 \text{Im} \left(e^{-i\pi\alpha} I_\xi(z) \right),$$

where $B_j = B_j(\kappa)$, $j = 1, 2$, are real constants. Recalling (4.16), this gives the following expression for $\mathcal{G}_\xi(z) = \mathcal{G}(z, -\xi, \xi)$:

$$\mathcal{G}(z, -\xi, \xi) = \frac{1}{\hat{c}} y^{\alpha + \frac{1}{\alpha} - 2} |z + \xi|^{1-\alpha} |z - \xi|^{1-\alpha} \operatorname{Im} (e^{-i\pi\alpha} I(z, -\xi, \xi)), \quad z \in \mathbb{H}, \quad \xi > 0,$$

where $\hat{c}(\kappa)$ is an overall real constant yet to be determined. Using translation invariance to extend this expression to an arbitrary starting point (ξ^1, ξ^2) , we find (2.8). The derivation here used that $4 < \kappa < 8$, but by analytic continuation we expect the same formula to hold for $0 < \kappa \leq 4$.

Remark 4.4. We remark here that the non-screened martingale obtained via Girsanov has the conformal dimensions

$$\lambda(z) = \lambda_*(z) = \frac{2-d}{2}, \quad \lambda_\infty = -\beta. \quad (4.25)$$

5 Schramm's formula

This section proves Theorem 2.1. The strategy is the same as in Schramm's original argument [35]. Assume $0 < \kappa < 8$, i.e., $\alpha = 8/\kappa > 1$. We write the function $\mathcal{M}(z, \xi)$ defined in (2.4) as

$$\mathcal{M}(z, \xi) = y^{\alpha-2} z^{-\frac{\alpha}{2}} (z - \xi)^{-\frac{\alpha}{2}} \bar{z}^{1-\frac{\alpha}{2}} (\bar{z} - \xi)^{1-\frac{\alpha}{2}} J(z, \xi), \quad z \in \mathbb{H}, \quad \xi > 0, \quad (5.1)$$

where $J(z, \xi)$ is defined by

$$J(z, \xi) = \int_{\bar{z}}^z (u - z)^\alpha (u - \bar{z})^{\alpha-2} u^{-\frac{\alpha}{2}} (u - \xi)^{-\frac{\alpha}{2}} du, \quad z \in \mathbb{H}, \quad \xi > 0, \quad (5.2)$$

and the contour from \bar{z} to z passes to the right of ξ as in Figure 1. We want to prove that the probability that the system started from $(0, \xi)$ passes to the right of z is given by

$$P(z, \xi) = \frac{1}{c_\alpha} \int_x^\infty \operatorname{Re} \mathcal{M}(x' + iy, \xi) dx', \quad x \in \mathbb{R}, \quad y > 0, \quad \xi > 0.$$

The idea is to apply Itô's formula and a stopping time argument to prove that the prediction is correct. In the present situation the hard work lies in verifying that the function $P(z, \xi)$ has sufficient regularity for Itô's formula to apply and that it satisfies the appropriate boundary conditions. Most of the needed analysis is carried out in Appendix A and the argument presented in this section uses results from there. Once we have proved Theorem 2.1, we easily obtain fusion formulas by simply collapsing the seeds.

5.1 Proof of Theorem 2.1

Let $P(z, \xi)$ be the function defined in (2.5). In Appendix A, we carefully analyze the function $P(z, \xi)$ and show that it is well-defined, smooth, and fulfills the correct boundary conditions. We summarize these facts here and then use them to give the short proof of Theorem 2.1.

Lemma 5.1. *The function $P(z, \xi)$ defined in (2.5) is a well-defined smooth function of $(z, \xi) \in \mathbb{H} \times (0, \infty)$.*

Proof. See Lemma A.8. □

Lemma 5.2. *We have*

$$|P(z, \xi)| \leq C (\arg z)^{\alpha-1}, \quad z \in \mathbb{H}, \quad \xi > 0, \quad (5.3a)$$

$$|P(z, \xi) - 1| \leq C (\pi - \arg z)^{\alpha-1}, \quad z \in \mathbb{H}, \quad \xi > 0. \quad (5.3b)$$

Proof. See Lemma A.12. □

Consider a system of commuting SLEs in \mathbb{H} started from 0 and $\xi > 0$, respectively. We grow both paths simultaneously with some given speeds $\lambda_j(t) \geq 0$. Write ξ_t^1 and ξ_t^2 for the Loewner driving terms of the system and let g_t denote the solution of (2.1) which uniformizes the whole system at capacity t . Then ξ_t^1 and ξ_t^2 are the images of the tips of the two curves under the conformal map g_t . Given a point $z \in \mathbb{H}$, let $Z_t = g_t(z)$ and let $\tau(z)$ denote the time that $\text{Im } g_t(z)$ first reaches 0.

Remark 5.3. A point $z \in \mathbb{H}$ lies to the left of both curves iff it lies to the left of the leftmost curve γ_1 starting from 0. Moreover, since the system is commuting, its distribution is independent of the order at which the two curves are grown. Hence we may as well assume that $\lambda_1(t) = 1$ and $\lambda_2(t) = 0$, but this assumption is not essential.

Lemma 5.4. *Let $z \in \mathbb{H}$. Define $P_t(z)$ by*

$$P_t(z) = P(Z_t - \xi_t^1, \xi_t^2 - \xi_t^1), \quad 0 \leq t < \tau(z).$$

Then $P_t(z)$ is a martingale for the system of commuting SLEs.

Proof. See Lemma A.13. □

Lemma 5.5. *Let $z \in \mathbb{H}$, and $\Theta_t^1 = \arg(Z_t - \xi_t^1)$. Then,*

$$\lim_{t \uparrow \tau(z)} \Theta_t^1 = 0 \quad \left(\text{resp.} \quad \lim_{t \uparrow \tau(z)} \Theta_t^1 = \pi \right),$$

if and only if z lies to the right (resp. left) of the curve γ_1 starting at 0.

Proof. See the proof of Lemma 3 in [35]. □

Lemma 5.6. *Let $\tilde{P}(z, \xi)$ be the probability that the point $z \in \mathbb{H}$ lies to the left of the two curves starting at 0 and $\xi > 0$, respectively. Then $\tilde{P}(z, \xi) = P(z, \xi)$, where $P(z, \xi)$ is the function defined in (2.5).*

Proof. By Lemma 5.5, the angle $\Theta_t^1 = \arg(Z_t - \xi_t^1)$ approaches π as $t \uparrow \tau(z)$ on the event that $z \in \mathbb{H}$ lies to the left of both curves. But (5.3b) shows that

$$|P_t(z) - 1| = |P(Z_t - \xi_t^1, \xi_t^2 - \xi_t^1) - 1| \leq C (\pi - \Theta_t^1)^{\alpha-1}, \quad z \in \mathbb{H}, \quad t \in [0, \tau(z)).$$

Consequently, on the event that z lies to the left of both curves, $P_t(z) \rightarrow 1$ as $t \uparrow \tau(z)$. A similar argument relying on (5.3a) shows that on the event that $z \in \mathbb{H}$ lies between or to the right of the two curves, then $P_t(z) \rightarrow 0$ as $t \uparrow \tau(z)$.

Let $\tau_n(z)$ be the stopping time defined by

$$\tau_n(z) = \inf \left\{ t \geq 0 : \sin \Theta_t^1 \leq \frac{1}{n} \right\}.$$

Since $P_t(z)$ is a martingale, we have

$$P_0(z) = \mathbf{E} \left[P_{\tau_n(z)}(z) \right], \quad n = 1, 2, \dots, \quad z \in \mathbb{H}.$$

By using the dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[P_{\tau_n(z)}(z) \right] = \tilde{P}(z, \xi).$$

Since $P_0(z) = P(z, \xi)$, this concludes the proof of the lemma and of Theorem 2.1. \square

5.2 The special case $\kappa = 4$

If $\alpha = \frac{8}{\kappa} > 1$ is an integer, the integral (5.2) defining $J(z, \xi)$ can be computed explicitly. However, the formulas quickly get very complicated as α increases. In this subsection, we consider the simplest case of $\alpha = 2$ (i.e. $\kappa = 4$). We remark that this case is particularly simple for one curve as well; indeed, the probability that an SLE_4 path passes to the right of z equals $(\arg z)/\pi$.

Proposition 5.7. *Let $\kappa = 4$. Then the function $P(z, \xi)$ in (2.5) is given explicitly by*

$$\begin{aligned} P(z, \xi) = \frac{1}{4\pi^2\xi} & \left\{ -2 \arctan \left(\frac{x}{y} \right) \left(\pi\xi - 2\xi \arctan \left(\frac{x-\xi}{y} \right) + 2y \right) \right. \\ & \left. + \pi^2\xi + (4y - 2\pi\xi) \arctan \left(\frac{x-\xi}{y} \right) \right\}, \quad z = x + iy \in \mathbb{H}, \quad \xi > 0. \end{aligned} \quad (5.4)$$

Proof. Let $\alpha = 2$. Then $c_\alpha = -2\pi^2$ and an explicit evaluation of the integral in (5.2) gives

$$J(z, \xi) = 2iy + \frac{2i}{\xi} ((z - \xi)^2 \arg(z - \xi) - z^2 \arg z), \quad z \in \mathbb{H}, \quad \xi > 0.$$

Using that

$$\arg z = \frac{\pi}{2} - \arctan \frac{x}{y} \quad \text{and} \quad \arg(z - \xi) = \frac{\pi}{2} - \arctan \frac{x - \xi}{y},$$

it follows that the function \mathcal{M} in (2.4) can be expressed as

$$\mathcal{M}(z, \xi) = \frac{2i}{z(z - \xi)\xi} \left\{ (z - \xi)^2 \left(\frac{\pi}{2} - \arctan \frac{x - \xi}{y} \right) - z^2 \left(\frac{\pi}{2} - \arctan \frac{x}{y} \right) + \xi y \right\}$$

for $z = x + iy \in \mathbb{H}$ and $\xi > 0$. Taking the real part of this expression and integrating with respect to x , we find that the function $P(z, \xi)$ in (2.5) is given by

$$\begin{aligned} P(z, \xi) &= \frac{1}{c_\alpha} \int_x^\infty \operatorname{Re} \mathcal{M}(x' + iy, \xi) dx' \\ &= \frac{1}{2\pi^2 \xi} \left\{ (2y - \pi\xi) \left(\frac{\pi}{2} - \arctan \frac{x' - \xi}{y} \right) - (\pi\xi + 2y) \left(\frac{\pi}{2} - \arctan \frac{x'}{y} \right) \right. \\ &\quad \left. - 2\xi \arctan \left(\frac{x'}{y} \right) \arctan \left(\frac{x' - \xi}{y} \right) \right\} \Big|_{x'=x}^\infty. \end{aligned}$$

The expression (5.4) follows. \square

Remark 5.8. In the fusion limit, equation (5.4) is consistent with the results of [20]. Indeed, in the limit $\xi \downarrow 0$ the expression (5.4) for $P(z, \xi)$ reduces to

$$P(z, 0+) = \frac{1}{4} - \frac{1}{\pi^2(1+t^2)} - \frac{\arctan t}{\pi} + \frac{(\arctan t)^2}{\pi^2}, \quad t := \frac{x}{y},$$

which is equation (25) in [20].

6 Green's function

In this section we prove Theorem 2.6. We recall from the discussion in Section 2 that the proof breaks down into proving Propositions 2.12 and 2.13. Proposition 2.12 establishes existence of a Green's function for $\operatorname{SLE}_\kappa(\rho)$ and provides a representation for this Green's function in terms of an expectation with respect to two-sided radial SLE. Proposition 2.13 then shows that the CFT prediction $\mathcal{G}_\xi(z)$ defined in (2.8) obeys this representation in the case of $\rho = 2$.

6.1 Existence of the Green's function: Proof of Proposition 2.12

Let $0 < \kappa \leq 4$ and $0 \leq \rho < 8 - \kappa$ and consider $\operatorname{SLE}_\kappa(\rho)$ started from (ξ^1, ξ^2) with $\xi^1 < \xi^2$. We recall our parameters

$$a = 2/\kappa, \quad r = \rho a/2 = \rho/\kappa, \quad \zeta(r) = \frac{r}{2a} (r + 2a - 1),$$

and the normalized local martingale

$$M_t^{(\rho)} = \left(\frac{\xi_t^2 - \xi_t^1}{\xi^2 - \xi^1} \right)^r g'_t(\xi^2)^{\zeta(r)}$$

by which we can weight $\operatorname{SLE}_\kappa$ in order to get $\operatorname{SLE}_\kappa(\rho)$, see Section 3. We will need a geometric regularity estimate. In order to state it, let $z \in \mathbb{H}$ and $0 < \epsilon_1 < \epsilon_2 < \operatorname{Im} z$. Let $\gamma : (0, 1] \rightarrow \mathbb{H}$ be a simple curve such that

$$\gamma(0+) = 0, \quad |\gamma(1) - z| = \epsilon_1, \quad |\gamma(t) - z| > \epsilon_1, \quad t \in [0, 1).$$

Write $H = \mathbb{H} \setminus \gamma$ where $\gamma = \gamma[0, 1]$. For $\epsilon > 0$ let \mathcal{B}_ϵ be the disk of radius ϵ about z and let U be the connected component containing z of $\mathcal{B}_{\epsilon_2} \cap H$. The set $\partial\mathcal{B}_{\epsilon_2} \cap \partial U$ consists of crosscuts of H . There is a unique one which separates z from ∞ in H and we denote this crosscut

$$\ell = \ell(z, \gamma, \epsilon_2). \quad (6.1)$$

See Figure 4.

Lemma 6.1. *Let $0 \leq \kappa \leq 4$. There exists $C < \infty$ and $\alpha > 0$ such that the following holds. Let $z \in \mathbb{H}$ and $0 < \epsilon_1 < \epsilon_2 < \text{Im } z$. For $\epsilon > 0$ define the stopping time*

$$\tau = \tau_\epsilon = \inf\{t \geq 0 : |\gamma(t) - z| \leq \epsilon\}.$$

If

$$\lambda = \lambda_{\epsilon_1, \epsilon_2} = \inf\{t \geq \tau_{\epsilon_1} : \gamma(t) \cap \ell \neq \emptyset\},$$

where

$$\ell = \ell(z, \gamma_{\tau_{\epsilon_1}}, \epsilon_2)$$

is as in (6.1), then on the event $\{\tau_{\epsilon_1} < \infty\}$, for $0 < \epsilon < \epsilon_1$,

$$\mathbf{P}\left(\lambda < \tau_\epsilon < \infty \mid \gamma_{\tau_{\epsilon_1}}\right) \leq C \left(\frac{\epsilon}{\epsilon_1}\right)^{2-d} \left(\frac{\epsilon_1}{\epsilon_2}\right)^{\beta/2},$$

where $\beta = 4a - 1$.

Proof. Write $\mathcal{B}_1 = \mathcal{B}(z, \epsilon_1)$, $\mathcal{B}_2 = \mathcal{B}(z, \epsilon_2)$ and $\tau_1 = \tau_{\epsilon_1}$, $\tau_2 = \tau_{\epsilon_2}$. Given γ_{τ_1} we consider the separating crosscut $\ell = \ell(z, \gamma_{\tau_1}, \epsilon_2)$. Let $\sigma = \max\{t \leq \tau_1 : \gamma(t) \in \ell\}$, which is not a stopping time but almost surely ℓ is a crosscut of H_σ which separates z from ∞ . Write V for the simply connected component containing z of $H_\sigma \setminus \ell$. Because one of the endpoints of ℓ is the tip $\gamma(\sigma)$, $g_\sigma(\partial V \setminus \ell) - W_\sigma$ is a bounded open interval I contained in either the positive or negative real axis. Almost surely, the curve $\gamma' = \gamma[\sigma, \lambda]$ is a crosscut of V starting and ending in ℓ . Note that $g_\sigma(\gamma') - W_\sigma$ is a curve in \mathbb{H} connecting 0 with $g_\sigma(\ell) - W_\sigma$, the latter which is a crosscut of \mathbb{H} separating I and $g_\sigma(z) - W_\sigma$ from ∞ in \mathbb{H} . Therefore, if $d = \text{dist}(\gamma_\lambda, z) \leq \epsilon_1$, we can use the Beurling estimate to see that

$$S_\lambda(z) \leq C \left(\frac{d}{\epsilon_2}\right)^{1/2}.$$

Consequently on the event that $\tau_1 < \infty$ and $d \geq 2\epsilon$, Lemma 3.1 shows that

$$\mathbf{P}(\tau_\epsilon < \infty \mid \gamma_\lambda) \leq C \left(\frac{\epsilon}{d}\right)^{2-d} \left(\frac{d}{\epsilon_2}\right)^{\beta/2} \leq C \left(\frac{\epsilon}{\epsilon_1}\right)^{2-d} \left(\frac{\epsilon_1}{\epsilon_2}\right)^{\beta/2}.$$

The last estimate uses that $\beta/2 - (2 - d) \geq 0$ when $\kappa \leq 4$ and that $d \leq \epsilon_1$.

On the event that $\tau_1 < \infty$ and $\epsilon < d \leq 2\epsilon$ we can use Lemma 3.2 (and the Beurling estimate to estimate the excursion measure) to see that

$$\mathbf{P}(\tau_\epsilon < \infty \mid \gamma_\lambda) \leq C \left(\frac{\epsilon}{\epsilon_2}\right)^{\beta/2} \leq C \left(\frac{\epsilon}{\epsilon_1}\right)^{2-d} \left(\frac{\epsilon_1}{\epsilon_2}\right)^{\beta/2};$$

the last estimate uses again that $\beta/2 - (2 - d) \geq 0$. □

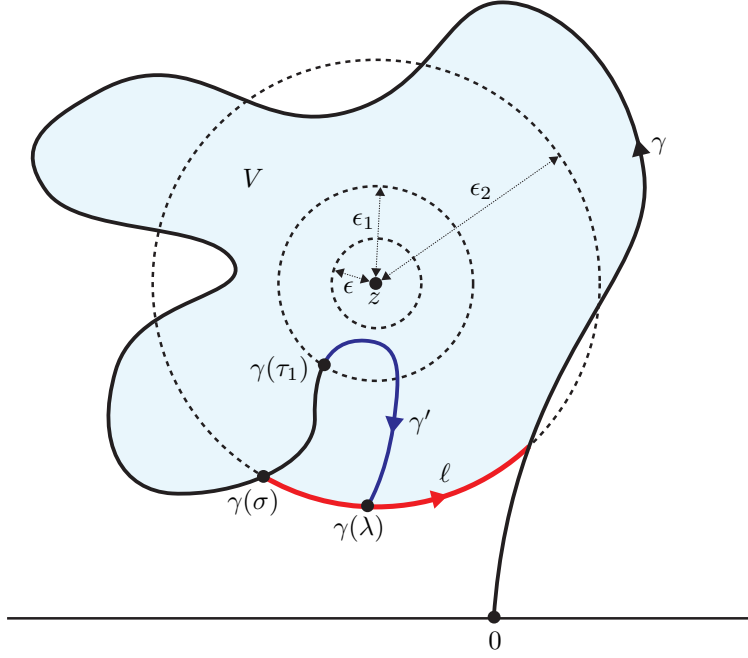


Figure 4. Schematic picture of the curve γ (solid), the open set V (shaded), and the crosscut ℓ defined in (6.1). If the path reenters V and hits $B(z, \epsilon)$ the “bad” event that $\lambda < \tau < \infty$ occurs. The probability of this event is estimated in Lemma 6.1.

Proof of Proposition 2.12. We may without loss of generality assume $\xi^1 = 0$ and $|z| = 1$. Constants are allowed to depend on z and ξ^2 . For $\epsilon > 0$, let

$$\tau = \tau_\epsilon = \inf\{s \geq 0 : \Upsilon_s \leq \epsilon\}, \quad \tau'_\epsilon = \inf\{s \geq 0 : |\gamma(s) - z| = \epsilon\},$$

and

$$\lambda = \inf\{s \geq \tau'_{\epsilon^{1/2}} : \gamma(t) \cap \ell \neq \emptyset\},$$

where $\ell = \ell(z, \gamma_{\tau'_{\epsilon^{1/2}}}, \epsilon^{1/4})$ is the separating crosscut as in Lemma 6.1; we are assuming ϵ is sufficiently small so that $\epsilon^{1/4} < \text{Im } z$. Let $E = E_\epsilon$ be the “good” event that $\tau < \lambda$.

We first claim that

$$\lim_{\epsilon \downarrow 0} \epsilon^{d-2} \mathbf{P}^\rho(\tau < \infty, E^c) = 0. \quad (6.2)$$

To see this, for $k = 1, \dots$,

$$\sigma_k = \inf_{t \geq 0} \{|\gamma(t)| \geq 2^k\},$$

and

$$U_k = \{\sigma_{k-1} \leq \tau < \sigma_k\}.$$

Using (3.7), we then have

$$\mathbf{P}^\rho(\tau < \infty, E^c) = \mathbf{E} \left[M_\tau^{(\rho)} 1_{\tau < \infty} 1_{E^c} \right] \leq \mathbf{P}(\tau < \infty, E^c) + C \sum_{k=1}^{\infty} 2^{kr} \mathbf{P}(\tau < \infty, E^c, U_k). \quad (6.3)$$

The first term on the right is $o(\epsilon^{2-d})$ using Lemma 6.1. We will estimate the series. For this, suppose $j = -1, \dots, J = \lceil \log_2(\epsilon^{-1}) \rceil$ and define

$$V_j^k = \{\tau'_{\epsilon 2^{j-1}} \leq \sigma_{k-1} < \tau'_{\epsilon 2^j}\}.$$

We can write

$$\mathbf{P}(\tau < \infty, E^c, U_k) = \sum_{j=-1}^J \mathbf{P}(\tau < \infty, E^c, U_k, V_j^k). \quad (6.4)$$

Let us first assume $j \leq \frac{1}{2} \log_2 \epsilon^{-1}$. We claim that on the event $V_j^k \cap \{\sigma_{k-1} < \infty\}$,

$$\mathbf{P}(\tau < \infty, E^c \mid \gamma_{\sigma_{k-1}}) \leq \mathbf{P}(\tau < \infty \mid \gamma_{\sigma_{k-1}}) \leq C \left(\frac{2^j \epsilon}{2^{2k}} \right)^{\beta/2} \left(\frac{\epsilon}{2^j \epsilon} \right)^{2-d}. \quad (6.5)$$

The first estimate is trivial and the second follows from Lemma 3.1 as follows. The curve $\gamma_{\sigma_{k-1}}$ is a crosscut of $D = 2^{k-1} \mathbb{D} \cap \mathbb{H}$ and so partitions D into exactly two components, one of which contains z . Consequently we get an upper bound on $S_{\sigma_{k-1}}(z)$ by estimating the probability of a Brownian motion from z to reach distance 2^{k-1} from 0 before hitting the real line or the curve. Thus, given the path up to time σ_{k-1} , the Beurling estimate shows that the probability that a Brownian motion starting at z reaches the circle of radius $2 \operatorname{Im} z$ about z without exiting $H_{\sigma_{k-1}}$ is $O((\epsilon 2^j / \operatorname{Im} z)^{1/2})$. Given this, the gambler's ruin estimate shows that the probability to reach modulus 2^{k-1} is $O(\operatorname{Im} z / 2^k)$. Hence, since $\operatorname{Im} z \leq 1$, we see that $S_{\sigma_{k-1}}^\beta(z) \leq c(\epsilon 2^j / 2^{2k})^{\beta/2}$. This gives (6.5). By Lemma 3.1 we have

$$\mathbf{P}(V_j^k \cap \{\sigma_{k-1} < \infty\}) \leq C(2^{j-1} \epsilon)^{2-d} S_0^\beta,$$

which combined with (6.5) gives

$$\sum_{j=-1}^{\lfloor \frac{1}{2} \log_2 \epsilon^{-1} \rfloor} \mathbf{P}(\tau < \infty, E^c, U_k, V_j^k) \leq C \epsilon^{2-d+\beta/4} 2^{-k\beta}. \quad (6.6)$$

It remains to handle the terms with $j > \frac{1}{2} \log_2 \epsilon^{-1}$ so that $2^j \epsilon > 2\epsilon^{1/2}$ which we now assume. Lemma 6.1 implies that there is $\alpha > 0$ such that on the event $\{\tau'_{\epsilon^{1/2}} < \infty\}$,

$$\mathbf{P}(\tau < \infty, E^c \mid \gamma_{\tau'_{\epsilon^{1/2}}}) \leq C \epsilon^{(2-d)/2+\alpha}.$$

Moreover, on the event $V_j^k \cap \{\sigma_{k-1} < \infty\}$, we can use Lemma 3.1 and the Beurling estimate as above to see that

$$\mathbf{P}(\tau'_{\epsilon^{1/2}} < \infty \mid \gamma_{\sigma_{k-1}}) \leq C \left(\frac{\epsilon^{1/2}}{2^j \epsilon} \right)^{2-d} \cdot \left(\frac{2^j \epsilon}{2^{2k}} \right)^{\beta/2}.$$

We conclude that

$$\mathbf{P}(\tau < \infty, E^c, U_k, V_j^k) \leq C \epsilon^{2-d+\alpha} (2^j \epsilon)^{\beta/2} 2^{-\beta k}.$$

So summing this over $j = \lceil \frac{1}{2} \log_2 \epsilon^{-1} \rceil, \dots, J$ and using also (6.6) shows that

$$2^{kr} \mathbf{P}(\tau < \infty, E^c, U_k) \leq 2^{k(r-\beta)} o(\epsilon^{2-d}).$$

Since $r - \beta < 0$ (equivalent to the condition $\rho < 8 - \kappa$) this is summable over k and the result is $o(\epsilon^{2-d})$. This proves (6.2).

Given (6.2) it is enough to prove that

$$\lim_{\epsilon \downarrow 0} \epsilon^{d-2} \mathbf{P}^\rho(\tau < \infty, E) = c_* G^{\kappa, \rho}(z, \xi^1, \xi^2).$$

For this let us fix $\epsilon > 0$ for the moment and recall that we write $\tau = \tau_\epsilon$. According to equation (3.14), we have

$$\mathbf{E}^*[f] = \tilde{G}_0^{-1} \mathbf{E}[\tilde{G}_t f], \quad t \geq 0,$$

whenever $f \in L^1(\mathbf{P}^*)$ is measurable with respect to $\tilde{\mathcal{F}}_t$. We change to the radial time parametrization and set $t = -\frac{1}{2a} \ln \epsilon$, so that $\epsilon = e^{-2at}$ and $s(t) = \tau_\epsilon$. Then $\tilde{G}_t = G_{\tau_\epsilon}$ and the function $M_{\tau_\epsilon}^{(\rho)} 1_{\tau_\epsilon < \infty} 1_E$ is measurable with respect to $\tilde{\mathcal{F}}_t = \mathcal{F}_{\tau_\epsilon}$, so we find

$$\mathbf{P}^\rho(\tau < \infty, E) = \mathbf{E} \left[M_{\tau_\epsilon}^{(\rho)} 1_{\tau_\epsilon < \infty} 1_E \right] = G_0 \mathbf{E}^* \left[G_{\tau_\epsilon}^{-1} M_{\tau_\epsilon}^{(\rho)} 1_{\tau_\epsilon < \infty} 1_E \right], \quad (6.7)$$

where G_0 is the SLE_κ Green's function. Thanks to the boundary conditions of the martingale G_t , we have $G_{s(t)} = G_\infty = 0$ on the event $\tau = \infty$. This means that $\mathbf{E}^*[1_{\tau=\infty}] = G_0^{-1} \mathbf{E}[G_\infty 1_{\tau=\infty}] = 0$. Hence we can remove the factor $1_{\tau < \infty}$ from the right-hand side of (6.7). Thus, using the definition (3.12) of G ,

$$\mathbf{P}^\rho(\tau < \infty, E) = \epsilon^{2-d} G_0 \mathbf{E}^* \left[M_{\tau_\epsilon}^{(\rho)} S_{\tau_\epsilon}^{-\beta} 1_E \right],$$

where $S_\tau = S_\tau(z)$. We need to show that

$$\lim_{\epsilon \downarrow 0} \mathbf{E}^* \left[M_{\tau_\epsilon}^{(\rho)} S_{\tau_\epsilon}^{-\beta} 1_E \right] = c_* \mathbf{E}^* \left[M_T^{(\rho)} \right], \quad \tau = \tau_\epsilon, \quad (6.8)$$

and where T is the time at which the path reaches z . Let $\tau'' = \tau_{\epsilon^{1/2}/4}$. Then $\tau'_{\epsilon^{1/2}} \leq \tau'' \leq \tau$ if ϵ is small enough. The Beurling estimate implies that

$$\left| M_{\tau_\epsilon}^{(\rho)} - M_{\tau''}^{(\rho)} \right| 1_E = O(\epsilon^{r/8}).$$

Using the invariant distribution (see Lemma 3.4) we have that $\mathbf{E}^* \left[S_{\tau_\epsilon}^{-\beta} \right] = O(1)$, so

$$\left| \mathbf{E}^* \left[\left(M_{\tau_\epsilon}^{(\rho)} - M_{\tau''}^{(\rho)} \right) S_{\tau_\epsilon}^{-\beta} 1_E \right] \right| \leq C \epsilon^{r/8} \mathbf{E}^* \left[S_{\tau_\epsilon}^{-\beta} \right] = O(\epsilon^{r/8}),$$

On the other hand, since $M_{\tau''}^{(\rho)} 1_{U_k} 1_{\tau < \infty} \leq C 2^{kr}$ the same argument that proved (6.2) shows that

$$\mathbf{E}^* \left[M_{\tau''}^{(\rho)} S_{\tau_\epsilon}^{-\beta} 1_{E^c} \right] = o(1)$$

as $\epsilon \rightarrow 0$. In other words,

$$\mathbf{E}^* \left[M_{\tau_\epsilon}^{(\rho)} S_{\tau_\epsilon}^{-\beta} 1_E \right] = \mathbf{E}^* \left[M_{\tau''}^{(\rho)} S_{\tau_\epsilon}^{-\beta} \right] + o(1).$$

Moreover,

$$\mathbf{E}^* \left[M_{\tau''}^{(\rho)} S_{\tau}^{-\beta} \right] = \mathbf{E}^* \left[M_{\tau''}^{(\rho)} \mathbf{E}^* \left[S_{\tau}^{-\beta} \mid \mathcal{F}_{\tau''} \right] \right].$$

Using Lemma 3.4 we see that there is $\alpha > 0$ such that

$$\mathbf{E}^* \left[S_{\tau}^{-\beta} \mid \mathcal{F}_{\tau''} \right] = \frac{c_*}{2} \int_0^{\pi} \sin \theta \, d\theta \, (1 + O(\epsilon^{\alpha})) = c_*(1 + O(\epsilon^{\alpha})).$$

It only remains to show that

$$\lim_{\epsilon \downarrow 0} \mathbf{E}^* \left[M_{\tau}^{(\rho)} \right] = \mathbf{E}^* \left[M_T^{(\rho)} \right].$$

For this we need to check that the sequence of integrands $(M_{\tau}^{(\rho)})$ is uniformly integrable as $\epsilon \downarrow 0$, that is, that for each $\epsilon_0 > 0$ there exists $R < \infty$ such that $\mathbf{E}^*[M_{\tau}^{(\rho)} 1_{M_{\tau}^{(\rho)} > R}] < \epsilon_0$ uniformly in ϵ . Since the only way in which $M_{\tau}^{(\rho)}$ can get large is by the path reaching a large diameter, this follows from a similar (but easier) argument as the one proving (6.2). We omit the details, but remark that this argument also needs $r - \beta < 0$. The Vitali convergence theorem now implies that $M_{\tau}^{(\rho)}$ converges to $M_T^{(\rho)}$ in $L^1(\mathbf{P}^*)$. The proof is complete. \square

6.2 Probabilistic representation for \mathcal{G}_{CFT} : Proof of Proposition 2.13

Let $0 < \kappa \leq 4$. Our goal is to show that

$$\mathcal{G}(z, \xi^1, \xi^2) = (\text{Im } z)^{d-2} \sin^{\beta}(\arg(z - \xi^1)) \mathbf{E}^*[M_T^{(2)}], \quad z \in \mathbb{H}, \quad \xi^1 < \xi^2, \quad (6.9)$$

where \mathbf{E}^* denotes expectation with respect to two-sided radial SLE_{κ} from ξ^1 through z , stopped at the hitting time T of z . Our first step is to use scale and translation invariance to reduce the relation (6.9), which depends on the four real variables $x = \text{Re } z, y = \text{Im } z, \xi^1, \xi^2$, to an equation involving only two independent variables.

6.2.1 The function $h(\theta^1, \theta^2)$

It follows from (2.7) and (2.8) that \mathcal{G} satisfies the scaling behavior

$$\mathcal{G}(\lambda z, \lambda \xi^1, \lambda \xi^2) = \lambda^{d-2} \mathcal{G}(z, \xi^1, \xi^2), \quad \lambda > 0.$$

Hence we can write

$$\mathcal{G}(z, \xi^1, \xi^2) = y^{d-2} \mathcal{H}(z, \xi^1, \xi^2),$$

where the function \mathcal{H} is homogeneous and translation invariant:

$$\mathcal{H}(\lambda z, \lambda \xi^1, \lambda \xi^2) = \mathcal{H}(z, \xi^1, \xi^2), \quad \lambda > 0, \quad (6.10a)$$

$$\mathcal{H}(z, \xi^1, \xi^2) = \mathcal{H}(x + \lambda, y, \xi^1 + \lambda, \xi^2 + \lambda), \quad \lambda \in \mathbb{R}. \quad (6.10b)$$

It follows that the value of $\mathcal{H}(x, y, \xi^1, \xi^2)$ only depends on the two angles θ^1 and θ^2 defined by

$$\theta^1 = \arg(z - \xi^1), \quad \theta^2 = \arg(z - \xi^2).$$

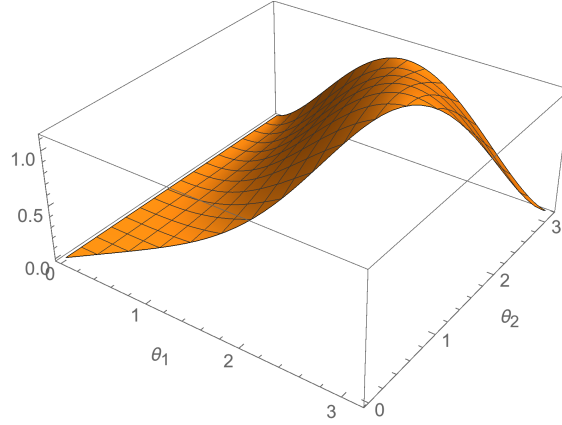


Figure 5. The graph of the function $h(\theta^1, \theta^2)$ for $\alpha = 5/2$.

In particular, if we let Δ denote the triangular domain

$$\Delta = \{(\theta^1, \theta^2) \in \mathbb{R}^2 \mid 0 < \theta^1 < \theta^2 < \pi\},$$

then we can define a function $h : \Delta \rightarrow \mathbb{R}$ for $\alpha \in (1, \infty) \setminus \mathbb{Z}$ by the equation

$$\mathcal{G}(z, \xi^1, \xi^2) = y^{d-2} h(\theta^1, \theta^2), \quad z \in \mathbb{H}, \quad -\infty < \xi^1 < \xi^2 < \infty. \quad (6.11)$$

According to Lemma 9.4, we can extend the definition of h to all $\alpha \in (1, \infty)$ by continuity. We write $h(\theta^1, \theta^2; \alpha)$ if we want to indicate the α -dependence of $h(\theta^1, \theta^2)$ explicitly. In terms of h , we can then reformulate equation (6.9) as follows:

$$h(\theta^1, \theta^2; \alpha) = (\sin^\beta \theta^1) \mathbf{E}^*[M_T^{(2)}], \quad (\theta^1, \theta^2) \in \Delta, \quad \beta \geq 1. \quad (6.12)$$

The following lemma, which is crucial for the proof of (6.12), describes the behavior of h near the boundary of Δ . In particular, it shows that $h(\theta^1, \theta^2)$ vanishes as θ^1 approaches 0 or π , and that the restriction of h to the top edge $\theta^2 = \pi$ of Δ equals $\sin^\beta \theta^1$. In other words, the lemma verifies that $\mathcal{G}(z, \xi^1, \xi^2)$ satisfies the appropriate boundary conditions.

Lemma 6.2 (Boundary behavior of h). *Let $\alpha \geq 2$. Then the function $h(\theta^1, \theta^2)$ defined in (6.11) is a smooth function of $(\theta^1, \theta^2) \in \Delta$ and has a continuous extension to the closure $\bar{\Delta}$ of Δ . This extension satisfies*

$$h(\theta^1, \pi) = \sin^\beta \theta^1, \quad \theta^1 \in [0, \pi], \quad (6.13)$$

$$h(\theta, \theta) = h_f(\theta), \quad \theta \in (0, \pi), \quad (6.14)$$

where $h_f(\theta)$ is defined in (8.8). Moreover, there exists a constant $C > 0$ such that

$$0 \leq h(\theta^1, \theta^2) \leq C \sin^\beta \theta^1, \quad (\theta^1, \theta^2) \in \bar{\Delta}, \quad (6.15)$$

and

$$\frac{|h(\theta^1, \theta^2) - h(\theta^1, \pi)|}{\sin^\beta \theta^1} \leq C \frac{|\pi - \theta^2|}{\sin \theta^1}, \quad (\theta^1, \theta^2) \in \Delta. \quad (6.16)$$

Proof. The rather technical proof involves asymptotic estimates of the integral in (2.7) and is given in Appendix B. \square

The derivation of formula (6.12) relies on an application of the optional stopping theorem to the martingale observable associated with \mathcal{G} . The following lemma gives an expression for this local martingale in terms of h .

Lemma 6.3 (Martingale observable for $\text{SLE}_\kappa(2)$). *Let $\theta_t^j = \arg(g_t(z) - \xi_t^j)$, $j = 1, 2$. Then*

$$\mathcal{M}_t = \Upsilon_t(z)^{d-2} h(\theta_t^1, \theta_t^2) \quad (6.17)$$

is a local martingale for $\text{SLE}_\kappa(2)$ started from (ξ^1, ξ^2) .

Proof. The proof follows from a direct computation using Itô's formula. In fact, since

$$\Upsilon_t^{d-2} h(\theta_t^1, \theta_t^2) = |g'_t(z)|^{2-d} \mathcal{G}(Z_t, \xi_t^1, \xi_t^2),$$

we see that \mathcal{M}_t is the martingale observable relevant for the Green's function found in Section 4 (cf. equation (4.14)). Hence, the result is a special case of Lemma 4.2 in the case when the growth speed $\lambda_2(t)$ of the second curve vanishes and the parameter σ_1, σ_2, s are given by (4.12). \square

Remark 6.4. The observable (6.17) is a local martingale also for the system of two commuting SLEs started from (ξ^1, ξ^2) by the same proof.

Let $z \in \mathbb{H}$ and consider $\text{SLE}_\kappa(2)$ started from (ξ^1, ξ^2) with $\xi^1 < \xi^2$. For each $\epsilon > 0$, we define the stopping time τ_ϵ by

$$\tau_\epsilon = \inf\{t \geq 0 : \Upsilon_t \leq \epsilon \Upsilon_0\},$$

where $\Upsilon_t = \Upsilon_t(z)$. Let $\epsilon > 0$ and $n \geq 1$. Then, since Υ_t is a nonincreasing function of t and $\Upsilon_0 = y$, we have

$$y \geq \Upsilon_{t \wedge n \wedge \tau_\epsilon} \geq \Upsilon_{\tau_\epsilon} = \epsilon y, \quad t \geq 0. \quad (6.18)$$

Hence, in view of the boundedness (6.15) of h , Lemma 6.3 implies that $(\mathcal{M}_{t \wedge \tau_\epsilon \wedge n})_{t \geq 0}$ is a true martingale for $\text{SLE}_\kappa(2)$. The optional stopping theorem therefore shows that

$$h(\theta^1, \theta^2) = \Upsilon_0^{2-d} \mathbf{E}^2[\Upsilon_{n \wedge \tau_\epsilon}^{d-2} h(\theta_{n \wedge \tau_\epsilon}^1, \theta_{n \wedge \tau_\epsilon}^2)]. \quad (6.19)$$

Recall that \mathbf{P} and \mathbf{P}^2 denote the SLE_κ and $\text{SLE}_\kappa(2)$ measures respectively, and that \mathbf{E} and \mathbf{E}^2 denote expectations with respect to these measures. Equations (3.7) and (3.8) imply

$$\mathbf{P}^2(V) = \mathbf{E}[M_t^{(2)} 1_V] \quad \text{for } V \in \mathcal{F}_t,$$

where

$$M_t^{(2)} = \left(\frac{\xi_t^2 - \xi_t^1}{\xi^2 - \xi^1} \right)^a g'_t(\xi^2)^{\frac{3a-1}{2}}.$$

In particular,

$$\mathbf{E}^2[f] = \mathbf{E}[M_{n \wedge \tau_\epsilon}^{(2)} f], \quad (6.20)$$

whenever $M_{n \wedge \tau_\epsilon}^{(2)} f$ is an $\mathcal{F}_{n \wedge \tau_\epsilon}$ -measurable $L^1(\mathbf{P})$ random variable.

Lemma 6.5. *For each $t \geq 0$, we have $M_t^{(2)} \in L^1(\mathbf{P})$.*

Proof. The identity

$$g'_t(\xi^2) = \exp \left(- \int_0^t \frac{ads}{(\xi_s^2 - \xi_s^1)^2} \right)$$

shows that

$$0 \leq g'_t(\xi^2) \leq 1, \quad t \geq 0. \quad (6.21)$$

Moreover, if E_t denotes the interval $E_t = (\xi_t^1, \xi_t^2) \subset \mathbb{R}$, then conformal invariance of harmonic measure gives

$$\lim_{s \rightarrow \infty} s\pi\omega(is, g_t^{-1}(E_t), \mathbb{H} \setminus \gamma_t) = \lim_{s \rightarrow \infty} s\pi\omega(g_t(is), E_t, \mathbb{H}) = |\xi_t^2 - \xi_t^1|.$$

Since the left-hand side is bounded above by a constant times $1 + \text{diam}(\gamma_t)$, this gives the estimate

$$|\xi_t^2 - \xi_t^1| \leq C(1 + \text{diam}(\gamma_t)), \quad t \geq 0. \quad (6.22)$$

On the other hand, since ξ_t^1 is the driving function for the Loewner chain g_t , we have (see e.g. Lemma 4.13 in [27])

$$\text{diam}(\gamma_t) \leq C \max \left\{ \sqrt{t}, \sup_{0 \leq s \leq t} |\xi_s^1| \right\}, \quad t \geq 0.$$

Combining the above estimates, we find

$$|M_t^{(2)}| \leq C |\xi_t^2 - \xi_t^1|^a \leq C(1 + \text{diam}(\gamma_t))^a \leq C \left(1 + \max \left\{ \sqrt{t}, \sup_{0 \leq s \leq t} |\xi_s^1| \right\} \right)^a, \quad t \geq 0.$$

Since ξ_t^1 is a \mathbf{P} -Brownian motion, it follows that $M_t^{(2)} \in L^1(\mathbf{P})$ for each $t \geq 0$. \square

As a consequence of (6.15), (6.18), and Lemma 6.5, the random variable

$$M_{n \wedge \tau_\epsilon}^{(2)} \Upsilon_{n \wedge \tau_\epsilon}^{d-2} h(\theta_{n \wedge \tau_\epsilon}^1, \theta_{n \wedge \tau_\epsilon}^2)$$

is $\mathcal{F}_{n \wedge \tau_\epsilon}$ -measurable and belongs to $L^1(\mathbf{P})$. Thus, we can use (6.20) to rewrite (6.19) as

$$h(\theta^1, \theta^2) = \Upsilon_0^{2-d} \mathbf{E}[M_{n \wedge \tau_\epsilon}^{(2)} \Upsilon_{n \wedge \tau_\epsilon}^{d-2} h(\theta_{n \wedge \tau_\epsilon}^1, \theta_{n \wedge \tau_\epsilon}^2)].$$

We split this into two terms depending on whether $\tau_\epsilon \leq n$ or $\tau_\epsilon > n$ as follows:

$$h(\theta^1, \theta^2) = \Upsilon_0^{2-d} \mathbf{E}[M_{\tau_\epsilon}^{(2)} \Upsilon_{\tau_\epsilon}^{d-2} h(\theta_{\tau_\epsilon}^1, \theta_{\tau_\epsilon}^2) 1_{\tau_\epsilon \leq n}] + F_{\epsilon, n}(\theta^1, \theta^2), \quad (6.23)$$

where

$$F_{\epsilon, n}(\theta^1, \theta^2) = \Upsilon_0^{2-d} \mathbf{E}[M_n^{(2)} \Upsilon_n^{d-2} h(\theta_n^1, \theta_n^2) 1_{\tau_\epsilon > n}].$$

We prove in Lemma 6.7 below that $F_{\epsilon, n}(\theta^1, \theta^2) \rightarrow 0$ as $n \rightarrow \infty$ for each fixed $\epsilon > 0$. Assuming this, we conclude from (6.23) that

$$h(\theta^1, \theta^2) = \Upsilon_0^{2-d} \lim_{n \rightarrow \infty} \mathbf{E}[M_{\tau_\epsilon}^{(2)} \Upsilon_{\tau_\epsilon}^{d-2} h(\theta_{\tau_\epsilon}^1, \theta_{\tau_\epsilon}^2) 1_{\tau_\epsilon \leq n}]. \quad (6.24)$$

Equations (3.12) and (3.14) imply

$$\mathbf{P}^*(V) = G_0^{-1} \mathbf{E}[G_t 1_V] \quad \text{for } V \in \mathcal{F}_t, \quad (6.25)$$

where $G_t = \Upsilon_t^{d-2} \sin^\beta \theta_t^1$. In particular,

$$\mathbf{E}[M_{\tau_\epsilon}^{(2)} f] = G_0 \mathbf{E}^*[G_{\tau_\epsilon}^{-1} M_{\tau_\epsilon}^{(2)} f],$$

whenever $M_{\tau_\epsilon}^{(2)} f$ is an $\mathcal{F}_{\tau_\epsilon}$ -measurable random variable in $L^1(\mathbf{P})$. Using Lemma 6.5 again, we see that the function $M_{\tau_\epsilon}^{(2)} \Upsilon_{\tau_\epsilon}^{d-2} h(\theta_{\tau_\epsilon}^1, \theta_{\tau_\epsilon}^2) 1_{\tau_\epsilon \leq n}$ is $\mathcal{F}_{\tau_\epsilon}$ -measurable and belongs to $L^1(\mathbf{P})$ for $n \geq 1$ and $\epsilon > 0$. Thus (6.24) can be expressed in terms of an expectation for two-sided radial SLE $_\kappa$ through z as follows:

$$\begin{aligned} h(\theta^1, \theta^2) &= \Upsilon_0^{2-d} G_0 \lim_{n \rightarrow \infty} \mathbf{E}^*[G_{\tau_\epsilon}^{-1} M_{\tau_\epsilon}^{(2)} \Upsilon_{\tau_\epsilon}^{d-2} h(\theta_{\tau_\epsilon}^1, \theta_{\tau_\epsilon}^2) 1_{\tau_\epsilon \leq n}] \\ &= \Upsilon_0^{2-d} G_0 \mathbf{E}^*[G_{\tau_\epsilon}^{-1} M_{\tau_\epsilon}^{(2)} \Upsilon_{\tau_\epsilon}^{d-2} h(\theta_{\tau_\epsilon}^1, \theta_{\tau_\epsilon}^2)], \end{aligned} \quad (6.26)$$

where the second equality is a consequence of dominated convergence and the fact that $\mathbf{E}^*[1_{\tau_\epsilon < \infty}] = 1$. Using that $G_t = \Upsilon_t^{d-2} \sin^\beta \theta_t^1$, we arrive at

$$h(\theta^1, \theta^2) = \sin^\beta \theta^1 \mathbf{E}^*[M_{\tau_\epsilon}^{(2)} \frac{h(\theta_{\tau_\epsilon}^1, \theta_{\tau_\epsilon}^2)}{\sin^\beta \theta_{\tau_\epsilon}^1}]. \quad (6.27)$$

In the limit as $\tau_\epsilon \rightarrow T$, where T denotes the hitting time of z , we have $\theta_{\tau_\epsilon}^2 \rightarrow \theta_T^2 = \pi$. Hence we use (6.13) to write (6.27) as

$$h(\theta^1, \theta^2) = \sin^\beta \theta^1 \mathbf{E}^*[M_{\tau_\epsilon}^{(2)}] + E(\theta^1, \theta^2),$$

where

$$E(\theta^1, \theta^2) = \sin^\beta \theta^1 \mathbf{E}^*\left[M_{\tau_\epsilon}^{(2)} \frac{h(\theta_{\tau_\epsilon}^1, \theta_{\tau_\epsilon}^2) - h(\theta_{\tau_\epsilon}^1, \pi)}{\sin^\beta \theta_{\tau_\epsilon}^1}\right].$$

But the estimate (6.16) implies

$$|E(\theta^1, \theta^2)| \leq c \mathbf{E}^*[M_{\tau_\epsilon}^{(2)} |\pi - \theta_{\tau_\epsilon}^2| (\sin \theta_{\tau_\epsilon}^1)^{-1}].$$

Equation (6.12) therefore follows from the following lemma.

Lemma 6.6. *For any $(\theta^1, \theta^2) \in \Delta$, it holds that*

$$\lim_{\epsilon \downarrow 0} \mathbf{E}^*[M_{\tau_\epsilon}^{(2)}] = \mathbf{E}^*[M_T^{(2)}] \quad (6.28)$$

and

$$\lim_{\epsilon \downarrow 0} \mathbf{E}^*[M_{\tau_\epsilon}^{(2)} |\pi - \theta_{\tau_\epsilon}^2| (\sin \theta_{\tau_\epsilon}^1)^{-1}] = 0. \quad (6.29)$$

Proof. For (6.28) it is enough to show that the family $\{M_{\tau_\epsilon}^{(2)}\}$ is uniformly integrable, i.e., that for every $\epsilon_0 > 0$, there exists an $R > 0$ such that

$$\mathbf{E}^* \left[M_{\tau_\epsilon}^{(2)} 1_{\{M_{\tau_\epsilon}^{(2)} > R\}} \right] < \epsilon_0 \quad (6.30)$$

for all small $\epsilon > 0$.

Let us prove (6.30). For $R > 0$, we define the stopping time λ_R by

$$\lambda_R = \inf\{t \geq 0 : |\gamma(t)| = R\}$$

and write $\{\tau_\epsilon < \infty\} = \cup_{j=0}^\infty E_j(\epsilon)$, where

$$E_j(\epsilon) = \{\lambda_{2^j} \leq \tau_\epsilon < \lambda_{2^{j+1}}\}.$$

The estimate (6.22) yields

$$|\xi_{\tau_\epsilon}^2 - \xi_{\tau_\epsilon}^1| \leq c 2^{j+1} \quad \text{on } E_j(\epsilon), \quad j \geq 0,$$

so, in view of (6.21), there exists a constant $c_0 > 0$ such that

$$|M_{\tau_\epsilon}^{(2)}| \leq c_0 2^{(j+1)a} \quad \text{on } E_j(\epsilon), \quad j \geq 0.$$

Here and below the constants C and c_0 are independent of $\epsilon > 0$, $j \geq 0$, and $R > 0$.

Suppose $R > 0$ and let $N := [a^{-1} \log_2(R/c_0)]$ denote the integer part of $a^{-1} \log_2(R/c_0)$. Then

$$|M_{\tau_\epsilon}^{(2)}| \leq c_0 2^{(j+1)a} \leq R \quad \text{on } E_j(\epsilon), \quad 0 \leq j \leq N-1.$$

Hence

$$\mathbf{E}^*[M_{\tau_\epsilon}^{(2)} 1_{\{M_{\tau_\epsilon}^{(2)} > R\}}] \leq \mathbf{E}^*[M_{\tau_\epsilon}^{(2)} 1_{\cup_{j=N}^\infty E_j(\epsilon)}] \leq c_0 \sum_{j=N}^\infty 2^{(j+1)a} \mathbf{E}^*[E_j(\epsilon)]. \quad (6.31)$$

The set $E_j(\epsilon)$ is $\mathcal{F}_{\tau_\epsilon}$ -measurable, so equation (6.25) gives

$$\mathbf{P}^*(E_j(\epsilon)) = \mathbf{P}^*(E_j(\epsilon) 1_{\tau_\epsilon < \infty}) = G_0^{-1} \mathbf{E}[G_{\tau_\epsilon} 1_{E_j(\epsilon)} 1_{\tau_\epsilon < \infty}] \leq C \epsilon^{d-2} \mathbf{E}[1_{E_j(\epsilon)} 1_{\tau_\epsilon < \infty}],$$

where we have used the following estimate in the last step:

$$|G_{\tau_\epsilon}| = (\Upsilon_0 \epsilon)^{d-2} \sin^\beta \theta_{\tau_\epsilon}^1 \leq C \epsilon^{d-2}.$$

We claim that

$$\mathbf{P}(E_j(\epsilon) \tau_\epsilon < \infty) \leq C \epsilon^{2-d} 2^{-j\beta} \quad \text{for } j > \log_2(4|z|). \quad (6.32)$$

Assuming for the moment that (6.32) holds, we find

$$\mathbf{P}^*(E_j(\epsilon)) \leq C 2^{-j\beta}, \quad j > \log_2(4|z|).$$

Employing this estimate in (6.31) we obtain

$$\mathbf{E}^*[M_{\tau_\epsilon}^{(2)} 1_{\{M_{\tau_\epsilon}^{(2)} > R\}}] \leq C \sum_{j=N}^\infty 2^{(j+1)a} 2^{-j\beta} \leq C \sum_{j=N}^\infty 2^{-j(3a-1)}$$

$$\leq C 2^{-N(3a-1)} \leq C \left(\frac{R}{c_0} \right)^{\frac{3a-1}{a}}, \quad \epsilon > 0, \quad N > \log_2(4|z|).$$

The condition $N > \log_2(4|z|)$ is fulfilled for all sufficiently large R . Hence, given $\epsilon_0 > 0$, by choosing R large enough, we can make $\mathbf{E}^*[M_{\tau_\epsilon}^{(2)} 1_{\{M_{\tau_\epsilon}^{(2)} > R\}}] < \epsilon_0$ for all $\epsilon > 0$. This proves (6.30), assuming (6.32) which we now verify. Suppose $j > \log_2(4|z|)$, i.e., $|z| < 2^j/4$. Let $D_j = \mathbb{H} \setminus \gamma([0, \lambda_{2^j}])$ and let $g_j : D_j \rightarrow \mathbb{H}$ be the uniformizing map with $g_j(\gamma(\lambda_{2^j})) = 0$. Let $k \geq 1$ be the unique integer such that $\frac{y}{2^k} < \Upsilon_{\lambda_{2^j}} \leq \frac{2y}{2^k}$. By (3.2), $\sin \arg(g_j(z))$ is bounded above by a constant times the probability that a Brownian motion starting at z reaches the circle of radius 2^j centered at the origin without leaving D_j . By a Beurling estimate, the probability that it reaches the circle of radius $2y$ centered at the origin without leaving D_j is bounded by $C 2^{-k/2}$, and given this the probability to reach the circle of radius 2^j is bounded by $C y 2^{-j}$. Hence

$$\sin \arg(g_j(z)) \leq C 2^{-k/2} y 2^{-j} \leq C \Upsilon_{\lambda_{2^j}}^{1/2} \sqrt{y} 2^{-j}. \quad (6.33)$$

On the other hand, by Lemma 3.1,

$$\mathbf{P}(E_j(\epsilon)) \leq C \left(\frac{\epsilon \Upsilon_0}{\Upsilon_{\lambda_{2^j}}} \right)^{2-d} \sin^\beta(\arg g_j(z)). \quad (6.34)$$

Combining (6.33) and (6.34), we find

$$\mathbf{P}(E_j(\epsilon)) \leq C \epsilon^{2-d} \left(\frac{\Upsilon_0}{\Upsilon_{\lambda_{2^j}}} \right)^{2-d} \Upsilon_{\lambda_{2^j}}^{\frac{\beta}{2}} y^{\frac{\beta}{2}} 2^{-j\beta} \leq C \epsilon^{2-d} 2^{-j\beta} \Upsilon_{\lambda_{2^j}}^{d-2+\frac{\beta}{2}}.$$

Since $d-2+\beta/2 \geq 0$ for $0 < \kappa \leq 4$, this proves (6.32). This completes the proof of (6.28).

It remains to prove (6.29). Note that there is a constant c (depending on z) such that

$$\mathbf{E}^*[M_{\tau_\epsilon}^{(2)} |\pi - \theta_{\tau_\epsilon}^2| (\sin \theta_{\tau_\epsilon}^1)^{-1}] \leq c \epsilon^{1/2} \mathbf{E}^*[M_{\tau_\epsilon}^{(2)} (\sin \theta_{\tau_\epsilon}^1)^{-1}].$$

Indeed, $|\pi - \theta_{\tau_\epsilon}^2|$ is bounded above by a constant times the harmonic measure from z in H_{τ_ϵ} of $[\xi^2, \infty)$, which by the Beurling estimate is $O(\epsilon^{1/2})$. On the other hand, recalling the definition of the measure \mathbf{P}^* and that $\beta - 1 \geq 0$ when $\kappa \leq 4$, we see that

$$\mathbf{E}^*[M_{\tau_\epsilon}^{(2)} (\sin \theta_{\tau_\epsilon}^1)^{-1}] = \frac{\epsilon^{d-2}}{\sin^\beta \theta^1} \mathbf{E}[M_{\tau_\epsilon}^{(2)} S_{\tau_\epsilon}^{\beta-1} 1_{\tau_\epsilon < \infty}] \leq \frac{\epsilon^{d-2}}{\sin^\beta \theta^1} \mathbf{E}[M_{\tau_\epsilon}^{(2)} 1_{\tau_\epsilon < \infty}].$$

Using Proposition 2.12 we see that last term converges, and is in particular bounded as $\epsilon \rightarrow 0$. This completes the proof, assuming Lemma 6.7. \square

Lemma 6.7. *Let*

$$F_{\epsilon,n}(\theta^1, \theta^2) = \Upsilon_0^{2-d} \mathbf{E}[M_n^{(2)} \Upsilon_n^{d-2} h(\theta_n^1, \theta_n^2) 1_{\tau_\epsilon > n}].$$

For each $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} F_{\epsilon,n}(\theta^1, \theta^2) = 0.$$

Proof. We can without loss in generality assume that $|z| \leq 1$. All constants are allowed to depend on z . Recall that

$$M_t^{(2)} = \left(\frac{\xi_t^2 - \xi_t^1}{\xi_t^2 - \xi_t^1} \right)^a g_t'(\xi^2)^{\frac{3a-1}{2}}.$$

We know that $|g_t'(\xi^2)|^{\frac{3a-1}{2}} \leq 1$, $\Upsilon_n^{d-2} 1_{\tau_\epsilon > n} \leq C\epsilon^{d-2}$, and $h(\theta_n^1, \theta_n^2) \leq C \sin^\beta \theta_n^1$, so it is enough to prove that

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[|\xi_n^2 - \xi_n^1|^a \sin^\beta \theta_n^1 \right] = 0.$$

For $k = 1, 2, \dots$, let U_k be the event that $2^k \sqrt{2an} \leq \text{rad } \gamma[0, n] \leq 2^{k+1} \sqrt{2an}$, where $\text{rad } K = \sup\{|z| : z \in K\}$. Since we have parametrized so that $\text{hcap } \gamma[0, t] = at$, we have $\mathbf{P}(\cup_{k \geq 0} U_k) = 1$. Fix $0 < p < 1/4$. For each integer $j \geq 0$, let V_j be the event that $2^j n^p \leq |\gamma(n)| \leq 2^{j+1} n^p$ and let V_{-1} be the event that $|\gamma(n)| < n^p$. (We could phrase these events in terms of stopping times.) We write

$$\mathbf{E} \left[|\xi_n^2 - \xi_n^1|^a \sin^\beta \theta_n^1 \right] \leq \sum_{k \geq 0} \mathbf{E} \left[|\xi_n^2 - \xi_n^1|^a \sin^\beta \theta_n^1 1_{U_k} \right]$$

For each k we will estimate

$$\mathbf{E} \left[|\xi_n^2 - \xi_n^1|^a \sin^\beta \theta_n^1 1_{U_k} \right] \leq \sum_{j=-1}^J \mathbf{E} \left[|\xi_n^2 - \xi_n^1|^a \sin^\beta \theta_n^1 1_{V_j} 1_{U_k} \right],$$

where $J = \lceil (\frac{1}{2} - p) \log_2(n) + k + \frac{1}{2} \log_2(2a) \rceil$. By Theorem 1.1 of [18] we have for $j = -1, 0, 1, \dots$,

$$\mathbf{P}(U_k \cap V_j) \leq C(n^{p-1/2} 2^{j-k})^\beta \mathbf{P}(U_k) \leq C(n^{p-1/2} 2^{j-k})^\beta. \quad (6.35)$$

On the event $V_{-1} \cap U_k$ we then use the trivial estimate $\sin \theta_n^1 \leq 1$ to find

$$\mathbf{E} \left[|\xi_n^2 - \xi_n^1|^a \sin^\beta \theta_n^1 1_{V_{-1}} 1_{U_k} \right] \leq C(2^k n^{1/2})^a \mathbf{P}(U_k \cap V_{-1}) \leq C(2^k n^{1/2})^a (n^{p-1/2} 2^{-k})^\beta.$$

Since $a - \beta < 0$ for $a > 1/3$ this is summable over k , and we see that

$$\mathbf{E} \left[|\xi_n^2 - \xi_n^1|^a \sin^\beta \theta_n^1 1_{V_{-1}} \right] \leq C n^{-((1/2-p)\beta - a/2)}. \quad (6.36)$$

When $j \geq 0$, we can estimate using the Beurling and Gambler's ruin estimates (see the previous lemma), to find

$$\sin^\beta \theta_n^1 1_{V_j} \leq C(n^{-p} 2^{-j})^\beta 1_{V_j},$$

and therefore

$$|\xi_n^2 - \xi_n^1|^a \sin^\beta \theta_n^1 1_{V_j} 1_{U_k} \leq C(n^{1/2} 2^k)^a (n^{-p} 2^{-j})^\beta 1_{V_j} 1_{U_k}.$$

So using (6.35),

$$\sum_{j=0}^J \mathbf{E} \left[|\xi_n^2 - \xi_n^1|^a \sin^\beta \theta_n^1 1_{V_j} 1_{U_k} \right] \leq C_p 2^{-k(\beta-a)} n^{-(\beta/2-a/2)} (\log_2 n + k).$$

This is summable over k when $a > 1/3$ for any choice of p and the sum is $o(1)$ as $n \rightarrow \infty$. Since $0 < p < 1/4$, the exponent in (6.36) is strictly negative whenever $a \geq 1/2$. The proof is complete. \square

7 Two paths getting near the same point: Proof of Lemma 2.9

This section proves the correlation estimate Lemma 2.9 and this will complete the proof of Theorem 2.6.

Lemma 7.1. *For any $\epsilon \in (0, 1/2)$, it holds that*

$$\partial_n \omega(-1, \partial B_\epsilon(0), \mathbb{D} \setminus (\overline{B_\epsilon(0)} \cup [0, 1])) \leq \frac{8\sqrt{\epsilon}}{\pi}.$$

Proof. Assume first that $\epsilon \in (0, 1)$. Let $z_1 = -1/\sqrt{z}$, where the branch cut for \sqrt{z} runs along \mathbb{R}_+ . The map $z \mapsto z_1$ takes $\Omega := \mathbb{D} \setminus (B_\epsilon(0) \cup [0, 1])$ onto $\Omega_1 := \{\rho e^{i\varphi} : \rho \in (1, \epsilon^{-1/2}), \varphi \in (0, \pi)\}$. The Joukowski transform $z_1 \mapsto z_2 := (z_1 + z_1^{-1})/2$ then maps Ω_1 conformally onto the semi-ellipse $\Omega_2 := E \cap \mathbb{H}$, where

$$E = \left\{ x + iy \mid \frac{x^2}{r_+^2} + \frac{y^2}{r_-^2} < 1 \right\}, \quad r_\pm = \frac{\epsilon^{-\frac{1}{2}} \pm \epsilon^{\frac{1}{2}}}{2}.$$

The composed map f defined by $f(z) = z_2$ is a conformal map $\Omega \rightarrow \Omega_2$ such that $f(-1) = 0$ and $f(\partial B_\epsilon(0)) = \mathbb{H} \cap \partial E$. Hence

$$\partial_n \omega(-1, \partial B_\epsilon(0), \Omega) = |f'(-1)| \partial_n \omega(0, \mathbb{H} \cap \partial E, \Omega_2).$$

For $z \in \mathbb{H} \cap B_{r_-}(0)$, we have the estimate

$$\omega(z, \mathbb{H} \cap \partial E, \Omega_2) \leq \omega(z, \mathbb{H} \cap \partial B_{r_-}(0), \mathbb{H} \cap B_{r_-}(0)) = \frac{2}{\pi} \left(\pi - \arg \left(\frac{z - r_-}{z + r_-} \right) \right),$$

where the explicit expression for the harmonic measure on the right-hand side can be verified by noting that $\arg(\frac{z-r_-}{z+r_-})$ is a harmonic function of z in the semidisk $\mathbb{H} \cap B_{r_-}(0)$ with boundary values $\pi/2$ and π on $\mathbb{H} \cap \partial B_{r_-}(0)$ and $[-r_-, r_-]$, respectively. Using that $f'(-1) = i/2$, this gives

$$\begin{aligned} \partial_n \omega(-1, \partial B_\epsilon(0), \Omega) &\leq |f'(-1)| \partial_n \omega(0, \mathbb{H} \cap \partial B_{r_-}(0), \mathbb{H} \cap B_{r_-}(0)) \\ &= \frac{1}{\pi} \frac{\partial}{\partial y} \Big|_{y=0} \left(\pi - \operatorname{Im} \ln \left(\frac{iy - r_-}{iy + r_-} \right) \right) = \frac{2}{\pi r_-}, \quad \epsilon \in (0, 1). \end{aligned}$$

Since $\frac{2}{\pi r_-} \leq 8\sqrt{\epsilon}/\pi$ for $\epsilon \in (0, 1/2)$, the lemma follows. \square

Lemma 7.2. *Let $z = x + iy \in \mathbb{H}$. Let $\epsilon > 0$ be given with $\epsilon < y/10$ and let $B := B_\epsilon(z)$. Let γ be a simple curve connecting 0 with the closed disc \bar{B} and staying in $\mathbb{H} \setminus \bar{B}$ except for the endpoints. Then*

$$\mathcal{E}_{\mathbb{H} \setminus (\gamma \cup \bar{B})}(\mathbb{R}_+, \partial B) \leq C(\epsilon/y)^{1/2},$$

where the constant $C < \infty$ is independent of γ , $z \in \mathbb{H}$, and $\epsilon \in (0, y/10)$.

Proof. Let $\varphi : \mathbb{H} \rightarrow \mathbb{D}$ be the conformal map with $\varphi(z) = 0$ and $\varphi(0) = 1$. Note that $|\varphi'(z)| = 1/(2y)$. Write γ', B' for the images of γ, B under φ and let $D' = \mathbb{D} \setminus (\gamma' \cup \overline{B'})$. By the distortion estimate (3.1), there is a $c > 0$ such that $B' \subset B'' := \{z : |z| < c(\epsilon/y)\}$. Monotonicity of harmonic measure and Beurling's projection theorem then imply that, for all $\zeta \in \partial\mathbb{D} \setminus \{1\}$,

$$\partial_n \omega(\zeta, \partial B'; D') \leq \partial_n \omega(-1, \partial B''; \mathbb{D} \setminus (\overline{B''} \cup [0, 1])).$$

But Lemma 7.1 shows that

$$\partial_n \omega(-1, \partial B''; \mathbb{D} \setminus (\overline{B''} \cup [0, 1])) \leq \frac{8\sqrt{c\epsilon/y}}{\pi}.$$

Hence

$$\mathcal{E}_{\mathbb{H} \setminus (\gamma \cup \overline{B})}(\mathbb{R}_+, \partial B) \leq \mathcal{E}_{D'}(\partial\mathbb{D} \setminus \{1\}, \partial B') \lesssim (\epsilon/y)^{1/2}.$$

□

We could quote Theorem 1.8 of [32] for a slightly different version of the next lemma, but since our proof is short and also that slightly different we will give it here.

Lemma 7.3. *Suppose $\kappa > 0$ and $\rho > \max\{-2, \kappa/2 - 4\}$ and consider $SLE_\kappa(\rho)$ started from $(0, 1)$. Let C_∞ denote the function C_t defined in (3.10) evaluated at $t = \infty$. Then there exists a $q > 0$ such that*

$$\mathbf{P}_{0,1}^\rho(C_\infty(1) \leq \epsilon) = \tilde{c} \epsilon^{\beta+\rho a} (1 + O(\epsilon^q)), \quad \epsilon \downarrow 0, \quad (7.1)$$

where the constant $\tilde{c} = \tilde{c}(\kappa, \rho)$ is given by

$$\tilde{c} = \frac{\Gamma(6a + a\rho)}{2a\Gamma(2a)\Gamma(4a + a\rho)}.$$

In particular, there is a constant $C < \infty$ such that

$$\mathbf{P}_{0,1}^\rho(\gamma \cap \eta \neq \emptyset) \leq C \mathcal{E}_{\mathbb{H} \setminus \eta}(\mathbb{R}_-, \eta)^{\beta+\rho a}$$

for all crosscuts η separating 1 from 0 in \mathbb{H} .

Proof. Write ξ_t^1 for the driving term of γ and let $\xi_t^2 = g_t(1)$, where g_t is the Loewner chain of γ . We get $SLE_\kappa(\rho)$ started from $(0, 1)$ by weighting SLE_κ by the local martingale (see (3.7))

$$M_t^{(\rho)} = (\xi_t^2 - \xi_t^1)^r g_t'(1)^{\zeta(r)}, \quad r = \rho a/2.$$

Let

$$N_t = C_t(1)^{-(\beta+a\rho)} A_t^{\beta+a\rho}, \quad A_t = \frac{\xi_t^2 - O_t}{\xi_t^2 - \xi_t^1}.$$

Direct computation shows that N_t is a local martingale for $SLE_\kappa(\rho)$ started from $(0, 1)$, which satisfies $N_0 = 1$. Moreover,

$$M_t^{(\rho)} N_t = M_t^{(\kappa-8-\rho)},$$

where $M_t^{(\kappa-8-\rho)}$ is the local SLE_κ martingale corresponding to the choice $r = r_\kappa(\kappa - 8 - \rho) = -\beta - \rho a/2$. We will work in the radial parametrization seen from 1. We set

$$s(t) = \inf\{s \geq 0 : C_s(1) = e^{-at}\}$$

and write $\hat{M}_t^\rho = M_{s(t)}^{(\rho)}$, etc. for the time-changed processes. We have

$$\begin{aligned} \mathbf{P}_{0,1}^\rho(s(t) < \infty) &= \mathbf{E} \left[\hat{M}_t^\rho 1_{s(t) < \infty} \right] \\ &= \mathbf{E} \left[\hat{M}_t^\rho \hat{N}_t \hat{N}_t^{-1} 1_{s(t) < \infty} \right] \\ &= e^{-a(\beta+a\rho)} \mathbf{E} \left[\hat{M}_t^{\kappa-8-\rho} \hat{A}_t^{-(\beta+a\rho)} 1_{s(t) < \infty} \right] \\ &= e^{-a(\beta+a\rho)} \mathbf{E}^* \left[\hat{A}_t^{-(\beta+a\rho)} 1_{s(t) < \infty} \right], \end{aligned}$$

where \mathbf{E}^* refers to expectation with respect to $\text{SLE}_\kappa(\kappa - 8 - \rho)$ started from $(0, 1)$. The exponent is positive if $\rho > \kappa/2 - 4$ and $\tilde{\rho} = \kappa - 8 - \rho < \kappa/2 - 4$. The key observation is that under the measure \mathbf{P}^* we have that $s(t) < \infty$ almost surely and that \hat{A}_t is positive recurrent and converges to an invariant distribution. This uses $\rho > \kappa/2 - 4$; see e.g., [1, 28]. In fact, we have the following formula for the limiting distribution (set $\nu = -r_\kappa(\kappa - 8 - \rho) = \beta + a\rho/2$ in Lemma 2.2 of [1]):

$$\pi(x) = c' x^{\beta+a\rho} (1-x)^{2a-1}, \quad c' = \frac{\Gamma(6a+a\rho)}{\Gamma(2a)\Gamma(4a+a\rho)};$$

It follows that

$$\mathbf{E}^* \left[\hat{A}_t^{-(\beta+a\rho)} 1_{s(t) < \infty} \right] = c' \int_0^1 (1-x)^{2a-1} dx \left(1 + O(e^{-qt}) \right),$$

which gives (7.1). By the distortion estimates (3.1), if $\tau_\epsilon = \inf\{t \geq 0 : \text{dist}(\gamma_t, 1) \leq \epsilon\}$, then

$$\mathbf{P}_{0,1}^\rho(\tau_\epsilon < \infty) \asymp \epsilon^{\beta+a\rho}.$$

The last assertion then follows using

$$\mathcal{E}_{\mathbb{H} \setminus \eta}(\mathbb{R}_-, \eta) \gtrsim \frac{\text{diam}(\eta)}{\text{dist}(0, \eta)} \wedge 1.$$

□

Lemma 7.4. *There is a constant $0 < c < \infty$ such that the following holds. Let D be a simply connected domain containing 0 and with three marked boundary points ζ, ξ, η . Suppose γ_ζ, γ_ξ are crosscuts of D which are disjoint except at η , and which connects η with ζ and η with ξ , respectively, and such that neither crosscut disconnects 0 from the other. Write D_ζ and D_ξ for the components of 0 of $D \setminus \gamma_\zeta$ and $D \setminus \gamma_\xi$ and let $D_{\zeta, \xi} = D_\zeta \cap D_\xi$. Suppose $0 < \epsilon < r/10$ and*

$$r_{D_\zeta}(0) \leq 4\epsilon, \quad r_{D_\xi}(0) \geq r. \tag{7.2}$$

Then if

$$r_{D_\zeta}(0) > \epsilon (1 + 3c\epsilon/r),$$

it holds that

$$r_{D_{\zeta, \xi}}(0) > \epsilon (1 + c\epsilon/r).$$

Proof. Let $\phi_\zeta : D_\zeta \rightarrow \mathbb{D}$ be the conformal map with $\phi_\zeta(0) = 0, \phi'_\zeta(0) > 0$. Note that $\phi_\zeta(\gamma_\xi)$ is a crosscut of \mathbb{D} . By distortion estimates, $\text{dist}(0, \partial D_\zeta) \asymp r_{D_\zeta}(0)$ while $\text{dist}(0, \gamma_\xi) \geq r_{D_\xi}(0)/4$. Therefore the Beurling estimate and the bounds in (7.2) show that there is a universal constant c_1 such that

$$\text{diam } \phi_\zeta(\gamma_\xi) \leq c_1(\epsilon/r)^{1/2}. \quad (7.3)$$

Write D' for the component containing 0 of $\mathbb{D} \setminus \phi_\zeta(\gamma_\xi)$. Note that $0 \in D'$ and let $\psi : D' \rightarrow \mathbb{D}$ with $\psi(0) = 0, \psi'(0) > 0$. Then the normalized Riemann map of $D_{\zeta, \xi}$ is $\psi \circ \phi_\zeta$ and so we have

$$r_{D_{\zeta, \xi}}(0) = r_{D_\zeta}(0) \psi'(0)^{-1}.$$

By (7.3), we have $\text{diam}(\mathbb{D} \setminus D') \leq c_2(\epsilon/r)^{1/2}$, so the logarithmic capacity of $\mathbb{D} \setminus D'$ is at most a universal constant c_3 times ϵ/r . Therefore,

$$1 \leq \psi'(0) \leq 1 + c_4\epsilon/r.$$

Hence

$$r_{D_{\zeta, \xi}}(0) \geq r_{D_\zeta}(0) \frac{1}{1 + c_4\epsilon/r}.$$

Consequently, if $r_{D_\zeta}(0) > \epsilon(1 + 3c_4\epsilon/r)$ we have

$$r_{D_{\zeta, \xi}}(0) > \epsilon(1 + c_4\epsilon/r).$$

□

Lemma 7.5. *There exists a constant $C < \infty$ such that if $0 < \epsilon < \delta < \text{Im } z/10$ and $\xi^1 < \xi^2$, then*

$$\mathbf{P}_{\xi_1, \xi_2} \left(\Upsilon_\infty^1(z) \leq \epsilon, \Upsilon_\infty^2(z) \leq \epsilon \right) \leq C(\epsilon/y)^{2-d+\beta/2+a}, \quad (7.4)$$

where $2 - d = 1 - 1/4a, \beta = 4a - 1$.

Remark. Notice that if $a \geq 1/2$, i.e., $\kappa \leq 4$, then $\beta/2 \geq 2 - d$, i.e., half the boundary exponent is larger than the bulk exponent, with strict inequality if $\kappa < 4$. Therefore, (7.4) implies that there for every $\kappa \leq 4$ is $u > 0$ such that

$$\mathbf{P}_{\xi_1, \xi_2} \left(\Upsilon_\infty^1(z) \leq \sqrt{\epsilon}, \Upsilon_\infty^2(z) \leq \sqrt{\epsilon} \right) = O_y(\epsilon^{2-d+u}). \quad (7.5)$$

Proof of Lemma 7.5. We first grow γ^1 starting from ξ^1 . The distribution is that of an $\text{SLE}_\kappa(2)$ started from (ξ_1, ξ_2) . Let τ_ϵ be the first time γ^1 hits the ball $B_\epsilon = B(z, 4\epsilon)$. By Proposition 2.12 we have

$$\mathbf{P}(\tau_\epsilon < \infty) \lesssim (\epsilon/y)^{2-d}. \quad (7.6)$$

On the event that $\tau_\epsilon < \infty$, we stop γ^1 at τ_ϵ and write $H_\epsilon^1 = \mathbb{H} \setminus \gamma_{\tau_\epsilon}^1$. Then almost surely, $C_\epsilon := \partial B_\epsilon \cap H_{\tau_\epsilon}^1$ is a crosscut of H_ϵ^1 . By Lemma 7.2 we have

$$\mathcal{E}_{H_\epsilon^1 \setminus C_\epsilon}([\xi_2, \infty), C_\epsilon) \lesssim (\epsilon/y)^{1/2}, \quad (7.7)$$

on the event that $\tau_\epsilon < \infty$, with a universal constant. Conditioned on $\gamma_{\tau_\epsilon}^1$ (after uniformizing H_ϵ^1) the distribution of γ^2 is that of $\text{SLE}_\kappa(2)$ started from $(\xi_{\tau_\epsilon}^2, \xi_{\tau_\epsilon}^1)$. We claim that on the event that $\tau_\epsilon < \infty$

$$\mathbf{P}\left(\gamma^2 \cap C_\epsilon \neq \emptyset \mid \gamma_{\tau_\epsilon}^1\right) \lesssim (\epsilon/y)^{\beta/2+a}. \quad (7.8)$$

Indeed, note that $g_{\tau_\epsilon}(C_\epsilon)$ is a crosscut of \mathbb{H} separating $\xi_{\tau_\epsilon}^1$ from $\xi_{\tau_\epsilon}^2$ and by (7.7) and conformal invariance

$$\mathcal{E}_{\mathbb{H} \setminus g_{\tau_\epsilon}(C_\epsilon)}([\xi_{\tau_\epsilon}^2, \infty), g_{\tau_\epsilon}(C_\epsilon)) \lesssim (\epsilon/y)^{1/2}.$$

The estimate (7.8) now follows from Lemma 7.3 with $\rho = 2$. We conclude the proof by combining (7.8) with (7.6). \square

We can now give the proof of Lemma 2.9.

Proof of Lemma 2.9. Consider a system of commuting SLEs started from $\xi^1 < \xi^2$. Then we want to prove that

$$\lim_{\epsilon \downarrow 0} \epsilon^{d-2} \mathbf{P}(\Upsilon_\infty(z) \leq \epsilon) = \lim_{\epsilon \downarrow 0} \epsilon^{d-2} \mathbf{P}(\Upsilon_\infty^1(z) \leq \epsilon) + \lim_{\epsilon \downarrow 0} \epsilon^{d-2} \mathbf{P}(\Upsilon_\infty^2(z) \leq \epsilon).$$

We can write

$$\begin{aligned} \mathbf{P}(\Upsilon_\infty(z) \leq \epsilon) &= \mathbf{P}(\Upsilon_\infty^1(z) \leq \epsilon) + \mathbf{P}(\Upsilon_\infty^2(z) \leq \epsilon) \\ &\quad - \mathbf{P}(\Upsilon_\infty^1(z) \leq \epsilon, \Upsilon_\infty^2(z) \leq \epsilon) \\ &\quad + \mathbf{P}(\Upsilon_\infty^1(z) > \epsilon, \Upsilon_\infty^2(z) > \epsilon, \Upsilon_\infty(z) \leq \epsilon). \end{aligned}$$

We know from Theorem 2.12 that the renormalized limits of the first two terms on the right exist. We will show that the remaining terms decay as $o(\epsilon^{2-d})$ and this will prove the lemma. For the third term the required estimate follows immediately from Lemma 7.5, so it remains to estimate the last term.

By distortion estimates we have that $\Upsilon_\infty \leq \epsilon$ implies $\text{dist}(\gamma^1 \cup \gamma^2, z) \leq 2\epsilon$. We may assume that $\text{dist}(\gamma^1, z) \leq 2\epsilon$, which in turn implies $\Upsilon_\infty^1(z) \leq 4\epsilon$. Using Lemma 7.4 with $r = \sqrt{\epsilon}$ we see that there is a constant c such that the following estimates hold:

$$\begin{aligned} \mathbf{P}(\Upsilon_\infty^1(z) > \epsilon, \Upsilon_\infty^2(z) > \epsilon, \Upsilon_\infty(z) \leq \epsilon) &\leq \mathbf{P}\left(\epsilon(1 + c\sqrt{\epsilon}) \leq \Upsilon_\infty^1(z) \leq 4\epsilon, \Upsilon_\infty^2(z) \leq \sqrt{\epsilon}\right) \\ &\quad + \mathbf{P}\left(\epsilon \leq \Upsilon_\infty^1(z) \leq \epsilon(1 + c\sqrt{\epsilon})\right) \\ &\leq \mathbf{P}\left(\Upsilon_\infty^1(z) \leq \sqrt{\epsilon}, \Upsilon_\infty^2(z) \leq \sqrt{\epsilon}\right) \\ &\quad + \mathbf{P}\left(\epsilon \leq \Upsilon_\infty^1(z) \leq \epsilon(1 + c\sqrt{\epsilon})\right). \end{aligned}$$

By (7.5) and the fact that $\beta/2 + a > 2 - d$,

$$\mathbf{P}\left(\Upsilon_\infty^1(z) \leq \sqrt{\epsilon}, \Upsilon_\infty^2(z) \leq \sqrt{\epsilon}\right) = O(\epsilon^{(2-d+\beta/2+a)/2}) = o(\epsilon^{2-d}).$$

We can use Theorem 2.12 to see that

$$\begin{aligned} \mathbf{P} \left(\epsilon \leq \Upsilon_{\infty}^1(z) \leq \epsilon(1 + c\sqrt{\epsilon}) \right) &= c_* G^{(2)}(z, \xi^1, \xi^2) \epsilon^{2-d} \left[(1 + c\sqrt{\epsilon})^{2-d} - 1 + o(1) \right] \\ &= o(\epsilon^{2-d}). \end{aligned}$$

(Again the error term depends on z and a .) This completes the proof. \square

8 Fusion

8.1 Schramm's formula

The function $P(z, \xi)$ in (2.5) extends continuously to $\xi = 0$; hence we obtain an expression for Schramm's formula in the fusion limit by simply setting $\xi = 0$ in the formulas of Theorem 2.1. In this way, we recover the formula of [20] and can give a rigorous proof of this formula.

Theorem 8.1 (Schramm's formula for fused $\text{SLE}_{\kappa}(2)$). *Let $0 < \kappa < 8$. Consider chordal $\text{SLE}_{\kappa}(2)$ started from $(0, 0+)$. Then the probability $P_f(z)$ that a given point $z = x + iy \in \mathbb{H}$ lies to the left of the curve is given by*

$$P_f(z) = \frac{1}{c_{\alpha}} \int_x^{\infty} \text{Re } \mathcal{M}_f(x' + iy) dx', \quad (8.1)$$

where $c_{\alpha} \in \mathbb{R}$ is the normalization constant in (2.6) and

$$\mathcal{M}_f(z) = y^{\alpha-2} z^{-\alpha} \bar{z}^{2-\alpha} \int_{\bar{z}}^z (u - z)^{\alpha} (u - \bar{z})^{\alpha-2} u^{-\alpha} du, \quad z \in \mathbb{H},$$

with the contour passing to the right of the origin. The function $P_f(z)$ can be alternatively expressed as

$$P_f(z) = \frac{\Gamma(\frac{\alpha}{2})\Gamma(\alpha)}{2^{2-\alpha}\pi\Gamma(\frac{3\alpha}{2}-1)} \int_{\frac{x}{y}}^{\infty} S(t') dt', \quad (8.2)$$

where the real-valued function $S(t)$ is defined by

$$\begin{aligned} S(t) &= (1 + t^2)^{1-\alpha} \left\{ {}_2F_1\left(\frac{1}{2} + \frac{\alpha}{2}, 1 - \frac{\alpha}{2}, \frac{1}{2}; -t^2\right) \right. \\ &\quad \left. - \frac{2\Gamma(1 + \frac{\alpha}{2})\Gamma(\frac{\alpha}{2})t}{\Gamma(\frac{1}{2} + \frac{\alpha}{2})\Gamma(-\frac{1}{2} + \frac{\alpha}{2})} {}_2F_1\left(1 + \frac{\alpha}{2}, \frac{3}{2} - \frac{\alpha}{2}, \frac{3}{2}; -t^2\right) \right\}, \quad t \in \mathbb{R}. \end{aligned}$$

Remark 8.2. Formula (8.2) for $P_f(z)$ coincides with equation (15) of [20].

Proof of Theorem 8.1. The expression (8.1) for $P_f(z)$ follows immediately by letting $\xi \rightarrow 0$ in (2.5). Since the right-hand side of (8.2) vanishes as $x \rightarrow \infty$, the representation (8.2) will follow if we can prove that

$$\frac{\Gamma(\frac{\alpha}{2})\Gamma(\alpha)}{2^{2-\alpha}\pi\Gamma(\frac{3\alpha}{2}-1)} S(x/y) = \frac{y}{c_{\alpha}} \text{Re } \mathcal{M}(x + iy, 0), \quad x \in \mathbb{R}, \quad y > 0, \quad \alpha > 1. \quad (8.3)$$

In order to prove (8.3), we write $\mathcal{M}_f = y^{\alpha-2} z^{-\alpha} \bar{z}^{2-\alpha} J_f(z)$, where $J_f(z)$ denotes the function $J(z, \xi)$ defined in (5.2) evaluated at $\xi = 0$, that is,

$$J_f(z) = \int_{\bar{z}}^z (u-z)^{\alpha} (u-\bar{z})^{\alpha-2} u^{-\alpha} du, \quad (8.4)$$

where the contour passes to the right of the origin. Let us first assume that $x > 0$. Then we can choose the vertical segment $[\bar{z}, z]$ as contour in (8.4). The change of variables $v = \frac{u-\bar{z}}{z-\bar{z}}$, which maps the segment $[z, \bar{z}]$ to the interval $[0, 1]$, yields

$$J_f(z) = e^{-i\pi\alpha} (z-\bar{z})^{2\alpha-1} z^{-\alpha} \int_0^1 v^{\alpha} (1-v)^{\alpha-2} \left(1 - v \frac{z-\bar{z}}{z}\right)^{-\alpha} dv, \quad x > 0,$$

where we have used that $(z - v(z - \bar{z}))^{-\alpha} = z^{-\alpha} (1 - v \frac{z-\bar{z}}{z})^{-\alpha}$ for $v \in [0, 1]$ and $x > 0$. The hypergeometric function ${}_2F_1$ can be defined for $w \in \mathbb{C} \setminus [0, \infty)$ and $0 < b < c$ by¹

$${}_2F_1(a, b, c; w) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 v^{b-1} (1-wv)^{-a} (1-v)^{c-b-1} dv.$$

This gives, for $x > 0$,

$$\mathcal{M}_f(z) = -iy^{3\alpha-3} z^{-2\alpha} \bar{z}^{2-\alpha} 2^{2\alpha-1} \frac{\Gamma(\alpha+1)\Gamma(\alpha-1)}{\Gamma(2\alpha)} {}_2F_1\left(\alpha, \alpha+1, 2\alpha; 1 - \frac{\bar{z}}{z}\right). \quad (8.5)$$

The argument $w = 1 - \frac{\bar{z}}{z}$ of ${}_2F_1$ in (8.5) crosses the branch cut $[1, \infty)$ for $x = 0$. Therefore, to extend the formula to $x \leq 0$, we need to find the analytic continuation of ${}_2F_1$. This can be achieved as follows. Using the general identities

$${}_2F_1(a, b, c; w) = {}_2F_1(b, a, c; w)$$

and (see [33, Eq. 15.8.13])

$${}_2F_1(a, b, 2b; w) = \left(1 - \frac{w}{2}\right)^{-a} {}_2F_1\left(\frac{a}{2}, \frac{a+1}{2}, b + \frac{1}{2}; \left(\frac{w}{2-w}\right)^2\right),$$

we can write the hypergeometric function in (8.5) as

$${}_2F_1\left(\alpha, \alpha+1, 2\alpha; 1 - \frac{\bar{z}}{z}\right) = \left(\frac{x}{z}\right)^{-\alpha-1} {}_2F_1\left(\frac{\alpha+1}{2}, \frac{\alpha}{2} + 1, \alpha + \frac{1}{2}; -t^{-2}\right), \quad x > 0, \quad (8.6)$$

where $t = x/y$. Using the identity (see [33, Eq. 15.8.2])

$$\begin{aligned} \frac{\sin(\pi(b-a))}{\pi} \frac{{}_2F_1(a, b, c; w)}{\Gamma(c)} &= \frac{(-w)^{-a}}{\Gamma(b)\Gamma(c-a)} \frac{{}_2F_1(a, a-c+1, a-b+1; \frac{1}{w})}{\Gamma(a-b+1)} \\ &\quad - \frac{(-w)^{-b}}{\Gamma(a)\Gamma(c-b)} \frac{{}_2F_1(b, b-c+1, b-a+1; \frac{1}{w})}{\Gamma(b-a+1)}, \quad w \in \mathbb{C} \setminus [0, \infty), \end{aligned}$$

¹Throughout the paper, we use the principal branch of ${}_2F_1$ which is defined and analytic for $w \in \mathbb{C} \setminus [1, \infty)$.

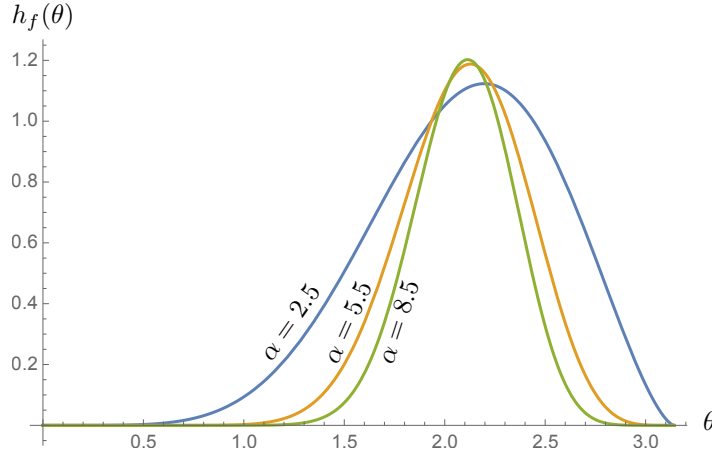


Figure 6. The graph of $h_f(\theta)$ for three different values of $\alpha = 8/\kappa$.

with $w = -t^{-2}$ to rewrite the right-hand side of (8.6), and substituting the resulting expression into (8.5), we find after simplification

$$\mathcal{M}_f(z) = - \frac{i\sqrt{\pi}2^{\alpha-1}\bar{z}\Gamma\left(\frac{\alpha-1}{2}\right)}{y^2\Gamma\left(\frac{\alpha}{2}\right)}S(t). \quad (8.7)$$

We have derived (8.7) under the assumption that $x > 0$, but since the hypergeometric functions in the definition of $S(t)$ are evaluated at the point $-t^2$ which avoids the branch cut for $z \in \mathbb{H}$, equation (8.7) is valid also for $x \leq 0$. Equation (8.3) is the real part of (8.7). \square

We obtain Schramm's formula for commuting SLE in the fusion limit as a corollary.

Corollary 8.3 (Schramm's formula for two fused commuting SLEs). *Let $0 < \kappa < 8$. Consider two fused commuting SLE_κ paths in \mathbb{H} started from 0 and growing toward infinity. Then the probability $P_f(z)$ that a given point $z = x + iy \in \mathbb{H}$ lies to the left of both curves is given by (8.1).*

Remark 8.4. We remark that the method adopted in [20] was based on exploiting so-called fusion rules, which produces a third order ODE for P_f which can then be solved in order to give the prediction in (8.2). However, even given the prediction (8.2) for P_f it is not clear how to proceed to give a proof that it is correct. As soon as the evolution starts, the tips of the curves are separated and the system leaves the fused state, so it seems difficult to apply a stopping time argument in this case.

8.2 Green's function

In this subsection, we derive an expression for the Green's function for $SLE_\kappa(2)$ started from $(0, 0+)$. Let $\alpha = 8/\kappa$. For $\alpha \in (1, \infty) \setminus \mathbb{Z}$, we define the 'fused' function $h_f(\theta) :=$

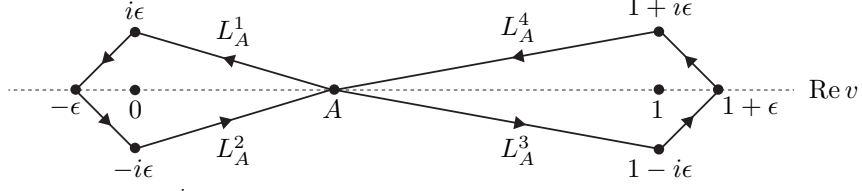


Figure 7. The contours L_A^j , $j = 1, \dots, 4$.

$h_f(\theta; \alpha)$ for $0 < \theta < \pi$ by

$$h_f(\theta) = \frac{\pi 2^{\alpha+1}}{\hat{c}} \sin\left(\frac{\pi\alpha}{2}\right) \sin^{2\alpha-2}(\theta) \operatorname{Re} \left[e^{-\frac{1}{2}i\pi\alpha} {}_2F_1\left(1-\alpha, \alpha, 1; \frac{1}{2}(1-i\cot(\theta))\right) \right], \quad (8.8)$$

where the constant $\hat{c} = \hat{c}(\kappa)$ is defined in (2.9). This definition is motivated by Lemma 6.2, which shows that $h_f(\theta)$ is the limiting value of $h(\theta^1, \theta^2)$ in the fusion limit $(\theta^1, \theta^2) \rightarrow (\theta, \theta)$. The next lemma shows that this definition of h_f can be extended by continuity to all $\alpha > 1$.

Given $A \in [0, 1]$ and $\epsilon > 0$ small, we let $L_A^j := L_A^j(\epsilon)$, $j = 1, \dots, 4$, denote the contours

$$\begin{aligned} L_A^1 &= [A, i\epsilon] \cup [i\epsilon, -\epsilon], & L_A^2 &= [-\epsilon, -i\epsilon] \cup [-i\epsilon, A], \\ L_A^3 &= [A, 1-i\epsilon] \cup [1-i\epsilon, 1+\epsilon], & L_A^4 &= [1+\epsilon, 1+i\epsilon] \cup [1+i\epsilon, A], \end{aligned} \quad (8.9)$$

oriented so that $\sum_1^4 L_A^j$ is a counterclockwise contour enclosing 0 and 1, see Figure 7.

Lemma 8.5. *For each $\theta \in (0, \pi)$, the function $h_f(\theta; \alpha)$ defined in (8.8) extends to a continuous function of $\alpha \in (1, \infty)$ satisfying*

$$h_f(\theta; n) = 2^{n-3} h_n \sin^{2n-2} \theta \times \begin{cases} \operatorname{Re} [2Y_2 - i\pi Y_1], & n = 2, 4, \dots, \\ \frac{2}{\pi} \operatorname{Re} [2iY_2 + \pi Y_1], & n = 3, 5, \dots, \end{cases} \quad (8.10)$$

where the constant $h_n \in \mathbb{C}$ is defined in (9.8) and the coefficients $Y_j := Y_j(\theta; n)$, $j = 1, 2$, are defined as follows: Introduce $y_j := y_j(v, \theta; n)$, $j = 0, 1$, by

$$\begin{aligned} y_0 &= v^{n-1} (1-vz)^{n-1} (1-v)^{-n}, \\ y_1 &= v^{n-1} (1-vz)^{n-1} (1-v)^{-n} (\ln v + \ln(1-vz) - \ln(1-v)), \end{aligned}$$

where $z = \frac{1-i\cot\theta}{2}$. Then

$$Y_1 = (2\pi i)^2 \operatorname{Res}_{v=1} y_0(v, \theta; n) \quad (8.11)$$

and

$$Y_2 = 2\pi i \int_{L_A^1 + L_A^2 + L_A^3 + L_A^4} y_1 dv + 2\pi^2 \int_{L_A^1 - L_A^2 - L_A^3 + L_A^4} y_0 dv, \quad (8.12)$$

where $1/z$ lies exterior to the contours and the principal branch is used for the complex powers throughout all integrations.

Proof. Let $n \geq 2$ be an integer. The standard hypergeometric function ${}_2F_1$ is defined by (see [33, Eq. 15.6.5])

$${}_2F_1(a, b, c; z) = \frac{e^{-c\pi i} \Gamma(c) \Gamma(1-b) \Gamma(1+b-c)}{4\pi^2} \times \int_A^{(0+, 1+, 0-, 1-)} v^{b-1} (1-vz)^{-a} (1-v)^{c-b-1} dv, \quad (8.13)$$

where $A \in (0, 1)$, $z \in \mathbb{C} \setminus [1, \infty)$, $b, c-b \neq 1, 2, 3, \dots$, and $1/z$ lies exterior to the contour. Hence, for $\alpha \notin \mathbb{Z}$ and $z \in \mathbb{C} \setminus [1, \infty)$,

$${}_2F_1(1-\alpha, \alpha, 1; z) = -\frac{1}{4\pi \sin(\pi\alpha)} Y(z; \alpha). \quad (8.14)$$

where

$$Y(z; \alpha) = \int_A^{(0+, 1+, 0-, 1-)} v^{\alpha-1} (1-vz)^{\alpha-1} (1-v)^{-\alpha} dv.$$

We first show that the function Y admits the expansion

$$Y(\theta; \alpha) = (\alpha - n)Y_1 + (\alpha - n)^2 Y_2 + O((\alpha - n)^3), \quad \alpha \rightarrow n, \quad (8.15)$$

where Y_1 and Y_2 are given by (8.11) and (8.12). Let $A \in (0, 1)$. Then

$$Y(z) = \left\{ (1 - e^{-2\pi i \alpha}) \int_{L_A^1} + e^{2\pi i(\alpha-1)} (1 - e^{-2\pi i \alpha}) \int_{L_A^2} + (e^{2\pi i(\alpha-1)} - 1) \int_{L_A^3} + e^{-2\pi i \alpha} (e^{2\pi i(\alpha-1)} - 1) \int_{L_A^4} \right\} v^{\alpha-1} (1-vz)^{\alpha-1} (1-v)^{-\alpha} dv.$$

Expansion around $\alpha = n$ gives (cf. the proof of (9.11)) equation (8.15) with Y_2 given by (8.12) and

$$Y_1 = 2\pi i \int_{L_A^1 + L_A^2 + L_A^3 + L_A^4} y_0 dv.$$

Since y_0 is analytic at $v = 0$ and has a pole at $v = 1$, we see that Y_1 can be expressed as in (8.11). This proves (8.15).

Equations (8.8) and (8.14) give

$$h_f(\theta) = -\frac{\pi 2^{\alpha+1}}{\hat{c}} \sin\left(\frac{\pi\alpha}{2}\right) \sin^{2\alpha-2}(\theta) \operatorname{Re} \left[\frac{e^{-\frac{\pi i \alpha}{2}}}{4\pi \sin(\pi\alpha)} Y(z; \alpha) \right]. \quad (8.16)$$

As $\alpha \rightarrow n$, we have

$$\hat{c}^{-1} = \begin{cases} -\frac{h_n}{(\alpha-n)^2} + \frac{a_n}{\alpha-n} + O(1), & n = 2, 4, \dots, \\ -\frac{h_n}{\alpha-n} + b_n + O(\alpha-n), & n = 3, 5, \dots, \end{cases}$$

where $a_n, b_n \in \mathbb{R}$ are real constants. We also have

$$2^{\alpha+1} = 2^{n+1} (1 + (\alpha - n) \ln 2 + O((\alpha - n)^2)),$$

$$\begin{aligned}
\sin\left(\frac{\pi\alpha}{2}\right) &= \begin{cases} \frac{(-1)^{\frac{n}{2}}\pi}{2}(\alpha-n) + O((\alpha-n)^3), & n=2,4,\dots, \\ (-1)^{\frac{n-1}{2}} + O((\alpha-n)^2), & n=3,5,\dots, \end{cases} \\
\sin^{2\alpha-2}\theta^1 &= \sin^{2n-2}(\theta^1)(1+2\ln(\sin\theta^1)(\alpha-n) + O((\alpha-n)^2)), \\
e^{-\frac{\pi i\alpha}{2}} &= e^{-\frac{\pi in}{2}}\left(1 - \frac{\pi i}{2}(\alpha-n) + O((\alpha-n)^2)\right), \\
\frac{1}{4\pi\sin(\pi\alpha)} &= \frac{(-1)^n}{4\pi^2(\alpha-n)} + O(\alpha-n),
\end{aligned}$$

Substituting the above expansions into (8.16) and using (8.15), we obtain, if $n \geq 2$ is even,

$$\begin{aligned}
h_f(\theta) &= \frac{2^{n-2}h_n \sin^{2n-2}(\theta^1) \operatorname{Re} Y_1}{\alpha-n} + 2^{n-3}h_n \sin^{2n-2}(\theta^1) \\
&\quad \times \operatorname{Re} \left[2Y_2 - i\pi Y_1 + 2\left(\ln 2 - \frac{a_n}{h_n} + 2\ln(\sin\theta^1)\right)Y_1 \right] + O(\alpha-n), \tag{8.17}
\end{aligned}$$

while, if $n \geq 2$ is odd,

$$\begin{aligned}
h_f(\theta) &= -\frac{2^{n-1}h_n \sin^{2n-2}(\theta^1) \operatorname{Im} Y_1}{\pi(\alpha-n)} + \frac{2^{n-2}h_n \sin^{2n-2}(\theta^1)}{\pi} \\
&\quad \times \operatorname{Re} \left[2iY_2 + \pi Y_1 + 2i\left(\ln 2 - \frac{b_n}{h_n} + 2\ln(\sin\theta^1)\right)Y_1 \right] + O(\alpha-n). \tag{8.18}
\end{aligned}$$

In order to establish (8.10), it is therefore enough to show that $\operatorname{Re} Y_1 = 0$ for even n and that $\operatorname{Im} Y_1 = 0$ for odd n .

Consider the function $J(z)$ defined by

$$J(z) = \int_{|v-1|=\epsilon} v^{n-1}(1-vz)^{n-1}(1-v)^{-n}dv, \quad z \in \mathbb{C} \setminus \{1\},$$

where $\epsilon > 0$ is so small that $1/z$ lies outside the contour. Then, by (9.16),

$$\overline{J(z)} = -\int_{|v-1|=\epsilon} v^{n-1}(1-v\bar{z})^{n-1}(1-v)^{-n}dv = -J(\bar{z}), \quad z \in \mathbb{C} \setminus \{1\}.$$

Letting $u = 1-v$, we can express $J(z)$ as

$$J(z) = -\int_{|u|=\epsilon} (1-u)^{n-1}(1-(1-u)z)^{n-1}u^{-n}du.$$

The change of variables $u = \frac{z-1}{z}\tilde{u}$ then yields

$$J(z) = (-1)^n \int_{|\tilde{u}|=\epsilon} (z - z\tilde{u} + \tilde{u})^{n-1}(1-\tilde{u})^{n-1}\tilde{u}^{-n}d\tilde{u} = (-1)^{n-1}J(1-z), \quad z \in \mathbb{C} \setminus \{0,1\}.$$

Hence, if $\operatorname{Re} z = 1/2$,

$$\overline{J(z)} = -J(\bar{z}) = -J(1-z) = (-1)^n J(z).$$

Since

$$Y_1(\theta; n) = 2\pi i J\left(\frac{1-i\cot\theta}{2}\right),$$

it follows that $\operatorname{Re} Y_1 = 0$ ($\operatorname{Im} Y_1 = 0$) for even (odd) n . This completes the proof of the lemma. \square

Taking $\xi \rightarrow 0+$ in Theorem 2.6, we obtain the following result for $SLE_\kappa(2)$ in the fusion limit.

Theorem 8.6 (Green's function for fused $SLE_\kappa(2)$). *Let $0 < \kappa \leq 4$ and consider chordal $SLE_\kappa(2)$ started from $(0, 0+)$. Then, for each $z = x + iy \in \mathbb{H}$,*

$$\lim_{\epsilon \rightarrow 0} \epsilon^{d-2} \mathbf{P}^2(\Upsilon_\infty(z) \leq \epsilon) = c_* \mathcal{G}_f(z), \quad (8.19)$$

where \mathbf{P}^2 is the $SLE_\kappa(2)$ measure, the function \mathcal{G}_f is defined by

$$\mathcal{G}_f(z) = (\operatorname{Im} z)^{d-2} h_f(\arg z), \quad z \in \mathbb{H}, \quad (8.20)$$

and the constant $c_* = c_*(\kappa)$ is given by (2.11).

For any given integer $n \geq 2$, we can compute the integrals in (8.11) and (8.12) defining Y_1 and Y_2 explicitly by taking the limit $\epsilon \rightarrow 0$. For the first few simplest cases $n = 2, 3, 4$, this leads to the expressions for the fused $SLE_\kappa(2)$ Green's function presented in the following proposition.

Proposition 8.7. *For $\alpha = 2, 3, 4$ (corresponding to $\kappa = 4, 8/3, 2$, respectively), the function $h_f(\theta)$ in (8.20) is given explicitly by*

$$h_f(\theta) = \begin{cases} \frac{2}{\pi}(\sin \theta - \theta \cos \theta) \sin \theta, & \alpha = 2, \\ \frac{8}{15\pi}(4\theta - 3 \sin 2\theta + 2\theta \cos 2\theta) \sin^2 \theta, & \alpha = 3, \quad 0 < \theta < \pi. \\ \frac{1}{12\pi}(27 \sin \theta + 11 \sin 3\theta - 6\theta(9 \cos \theta + \cos 3\theta)) \sin^3 \theta, & \alpha = 4, \end{cases}$$

Proof. The proof relies on long but straightforward computations and is similar to that of Proposition 9.5. \square

Remark 8.8. The formulas in Proposition 8.7 can also be obtained by taking the limit $\theta^2 \downarrow \theta^1$ in the formulas of Proposition 9.5.

In view of Lemma 2.9, it follows from Theorem 2.6 that the Green's function for two fused commuting SLEs started from 0 is given by the symmetrized expression $\mathcal{G}_f(z) + \mathcal{G}_f(-\bar{z})$. We formulate this as a corollary.

Corollary 8.9 (Green's function for two fused commuting SLEs). *Let $0 < \kappa \leq 4$. Consider a system of two fused commuting SLE_κ paths in \mathbb{H} started from 0 and growing towards ∞ . Then, for each $z = x + iy \in \mathbb{H}$,*

$$\lim_{\epsilon \rightarrow 0} \epsilon^{d-2} \mathbf{P}(\Upsilon_\infty(z) \leq \epsilon) = c_*(\mathcal{G}_f(z) + \mathcal{G}_f(-\bar{z})),$$

where $d = 1 + \kappa/8$, the constant $c_* = c_*(\kappa)$ is given by (2.11), and the function \mathcal{G}_f is defined by (8.20).

9 The function $\mathcal{G}(z, \xi^1, \xi^2)$ when α is an integer

In Section 2, we defined the function $\mathcal{G}(z, \xi^1, \xi^2)$ for noninteger values of $\alpha = 8/\kappa > 1$ by equation (2.8). We then claimed that \mathcal{G} can be extended to integer values of α by continuity. The purpose of this section is to verify this claim and to provide formulas for $\mathcal{G}(z, \xi^1, \xi^2)$ in the case when α is an integer.

9.1 A representation for h

Equations (2.7) and (2.8) express \mathcal{G} in terms of an integral with a Pochhammer contour enclosing the variable points ξ^2 and z . In order to perform asymptotic calculations (and in order to make contact with the theory of hypergeometric integrals), it is convenient to express \mathcal{G} in terms of an integral whose contour encloses the fixed points 0 and 1. This can easily be achieved by means of a linear fractional transformation which maps z and ξ^2 to 0 and 1, respectively. Moreover, instead of considering \mathcal{G} directly, it is convenient to work with the associated scale invariant function $h(\theta^1, \theta^2)$ defined in (6.11).

Lemma 9.1 (Representation for h). *Define the function $F(w_1, w_2)$ by*

$$F(w_1, w_2) = \int_A^{(0+, 1+, 0-, 1-)} v^{\alpha-1} (v - w_1)^{\alpha-1} (v - w_2)^{-\frac{\alpha}{2}} (1 - v)^{-\frac{\alpha}{2}} dv, \quad w_1, w_2 \in \mathbb{C} \setminus [0, \infty), \quad (9.1)$$

where $A \in (0, 1)$ is a basepoint and w_1, w_2 are assumed to lie outside the contour. For each noninteger $\alpha > 1$, the function h defined in (6.11) admits the representation

$$h(\theta^1, \theta^2; \alpha) = \frac{\sin^{\alpha-1} \theta^1}{\hat{c}} \operatorname{Im} \left[\sigma(\theta^2) (-e^{i\theta^2})^{\alpha-1} F(w_1, w_2) \right], \quad (\theta^1, \theta^2) \in \Delta, \quad (9.2)$$

where w_1 and w_2 are given by

$$w_1 := 1 - e^{-2i\theta^2}, \quad w_2 := \frac{1 - e^{-2i\theta^2}}{1 - e^{-2i\theta^1}} = \frac{\sin \theta^2}{\sin \theta^1} e^{-i(\theta^2 - \theta^1)}, \quad (9.3)$$

the constant \hat{c} is defined in (2.9), and

$$\sigma(\theta^2) = \begin{cases} e^{-i\pi\alpha}, & \theta^2 \geq \frac{\pi}{2}, \\ e^{i\pi\alpha}, & \theta^2 < \frac{\pi}{2}. \end{cases} \quad (9.4)$$

Remark 9.2. For $(\theta^1, \theta^2) \in \Delta$, w_1 and w_1/w_2 lie on the circle of radius one centered at 1 (see Figure 8), while w_2 lies in the open lower half-plane, i.e., $\operatorname{Im} w_2 < 0$.

Remark 9.3. The value of $F(w_1, w_2)$ in (9.2) is, strictly speaking, not well-defined by (9.1) for $\theta^2 = \pi/2$, because in this case $w_2 = 2$. However, by analytic continuation, the function F in (9.2) extends to a multiple-valued function of $w_1, w_2 \in \mathbb{C} \setminus \{0, 1\}$. Equation (9.2) then extends continuously across the line $\theta^2 = \pi/2$.

Proof. Introducing the new variable $v = \frac{u-z}{\xi^2-z}$ in (2.7), we find

$$\begin{aligned} I(z, \xi^1, \xi^2) &= \int_A^{(0+, 1+, 0-, 1-)} (v(\xi^2 - z))^{\alpha-1} (z - \bar{z} + v(\xi^2 - z))^{\alpha-1} \\ &\quad \times (z + v(\xi^2 - z) - \xi^1)^{-\frac{\alpha}{2}} ((\xi^2 - z)(1 - v))^{-\frac{\alpha}{2}} (\xi^2 - z) dv \\ &= (\xi^2 - z)^{\alpha-1} F(w_1, w_2) \times \begin{cases} 1, & x \leq \xi^2, \\ e^{2i\pi(\alpha-1)}, & x > \xi^2, \end{cases} \end{aligned} \quad (9.5)$$

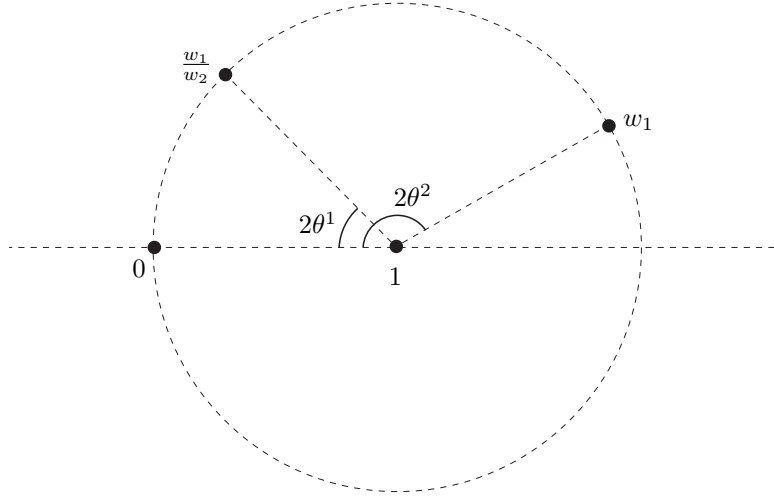


Figure 8. The complex numbers $w_1 = 1 - e^{-2i\theta^2}$ and $w_1/w_2 = 1 - e^{-2i\theta^1}$ lie on the circle of radius one centered at 1.

where $\frac{\xi+z}{z-\xi}$ and $\frac{z-\bar{z}}{z-\xi}$ are not enclosed by the contour, and the variables

$$w_1 = \frac{z - \bar{z}}{z - \xi^2} = \frac{2i}{\cot \theta^2 + i}, \quad w_2 = \frac{\xi^1 - z}{\xi^2 - z} = \frac{\cot \theta^1 + i}{\cot \theta^2 + i},$$

can be expressed as in (9.3). The extra factor of $e^{2i\pi(\alpha-1)}$ in (9.5) which is present for $x > \xi^2$ arises from the factor $(z - \bar{z} + v(\xi^2 - z))^{\alpha-1}$ as follows. Let v belong to the contour in (9.5). Then the complex number $z - \bar{z} + v(\xi^2 - z)$ lies in the upper half-plane. If $\frac{\pi}{2} \leq \theta^2 < \pi$ (i.e. if $x \leq \xi^2$), then $v - w_1$ also lies in the upper half-plane, but if $0 < \theta^2 < \pi/2$ (i.e. if $x > \xi^2$), then $v - w_1$ has crossed the negative real axis into the lower half-plane. The factor $e^{2\pi i(\alpha-1)}$ is inserted to compensate for this crossing of the branch cut.

Equations (2.8), (6.11), and (9.5) give

$$\begin{aligned} h(\theta^1, \theta^2) &= \frac{1}{\hat{c}} y^{\alpha-1} |z - \xi^1|^{1-\alpha} |z - \xi^2|^{1-\alpha} |z - \xi^1|^{1-\alpha} |z - \xi^2|^{1-\alpha} \text{Im} [e^{-i\pi\alpha} I(z, \xi^1, \xi^2)] \\ &= \frac{1}{\hat{c}} \sin^{\alpha-1}(\theta^1) \sin^{\alpha-1}(\theta^2) \text{Im} [\sigma(\theta^2)(-\cot \theta^2 - i)^{\alpha-1} F(w_1, w_2)], \end{aligned} \quad (9.6)$$

where σ is given by (9.4). The representation (9.2) follows. \square

Let $1_{\{\theta^2 \geq \frac{\pi}{2}\}}$ denote the function which equals 1 if $\theta^2 \geq \pi/2$ and 0 otherwise. Let $L_A^j := L_A^j(\epsilon)$, $j = 1, \dots, 4$, be the contours defined in (8.9).

Lemma 9.4. For each $(\theta^1, \theta^2) \in \Delta$, $h(\theta^1, \theta^2; \alpha)$ extends to a continuous function of

$\alpha \in (1, \infty)$ such that

$$h(\theta^1, \theta^2; n) = h_n \sin^{\alpha-1} \theta^1 \times \begin{cases} \operatorname{Im} [e^{(n-1)i\theta^2} (F_2 + i(\theta^2 - 2\pi 1_{\{\theta^2 \geq \frac{\pi}{2}\}}) F_1)], & n = 2, 4, \dots, \\ \operatorname{Im} [e^{(n-1)i\theta^2} F_1], & n = 3, 5, \dots, \end{cases} \quad (9.7)$$

where the constant $h_n \in \mathbb{R}$ is defined by

$$h_n = \begin{cases} -\frac{i^n \Gamma(\frac{n}{2}) \Gamma(n)}{2\pi^3 \Gamma(\frac{3n}{2}-1)}, & n = 2, 4, \dots, \\ -\frac{i^{n+1} \Gamma(\frac{n}{2}) \Gamma(n)}{4\pi^2 \Gamma(\frac{3n}{2}-1)}, & n = 3, 5, \dots, \end{cases} \quad (9.8)$$

and the coefficients $F_j := F_j(\theta^1, \theta^2; n)$, $j = 1, 2$, are defined as follows: Let w_1 and w_2 be given by (9.3) and define $f_j := f_j(v, \theta_1, \theta_2; n)$, $j = 0, 1$, by

$$\begin{aligned} f_0 &= v^{n-1} (v - w_1)^{n-1} (v - w_2)^{-\frac{n}{2}} (1 - v)^{-\frac{n}{2}}, \\ f_1 &= v^{n-1} (v - w_1)^{n-1} (v - w_2)^{-\frac{n}{2}} (1 - v)^{-\frac{n}{2}} \left(\ln v + \ln(v - w_1) - \frac{\ln(v - w_2)}{2} - \frac{\ln(1 - v)}{2} \right). \end{aligned}$$

Then

$$F_1 = \begin{cases} \frac{4\pi^2 (-1)^{\frac{n}{2}+1}}{(\frac{n}{2}-1)!} \partial_v^{\frac{n}{2}-1} \Big|_{v=1} (v^{n-1} (v - w_1)^{n-1} (v - w_2)^{-\frac{n}{2}}), & n = 2, 4, \dots, \\ 2\pi i \int_{L_0^3(\epsilon) - L_0^4(\epsilon)} f_0 dv, & n = 3, 5, \dots \end{cases} \quad (9.9)$$

and

$$F_2 = -2\pi^2 \int_{L_0^3(\epsilon)} f_0 dv + 2\pi i \int_{|v-1|=\epsilon} f_1 dv, \quad n = 2, 4, \dots, \quad (9.10)$$

where $\epsilon > 0$ is so small that w_1, w_2 lie exterior to the contours and the principal branch is used for all complex powers in the integrals.

Proof. Let $n \geq 2$ be an integer. We first show that the function F defined in (9.1) admits the expansion

$$F(\theta^1, \theta^2; \alpha) = (\alpha - n) F_1 + (\alpha - n)^2 F_2 + O((\alpha - n)^3), \quad \alpha \rightarrow n, \quad (9.11)$$

where F_1 and F_2 are given by (9.9) and (9.10). Define $f := f(v, w_1, w_2; \alpha)$ by

$$f = v^{\alpha-1} (v - w_1)^{\alpha-1} (v - w_2)^{-\frac{\alpha}{2}} (1 - v)^{-\frac{\alpha}{2}}.$$

Let $\epsilon > 0$ be small and fix $A \in (\epsilon, 1 - \epsilon)$. Then we can rewrite (9.1) as

$$\begin{aligned} F(w_1, w_2) &= \left(\int_{L_A^1 - L_A^3} + e^{2\pi i \alpha} \int_{L_A^2 + L_A^3} + e^{\pi i \alpha} \int_{L_A^4 - L_A^2} + e^{-\pi i \alpha} \int_{-L_A^1 - L_A^4} \right) f(v) dv \\ &= (1 - e^{-\pi i \alpha}) \left(\int_{L_A^1} + e^{2\pi i \alpha} \int_{L_A^2} \right) f(v) dv + (e^{\pi i \alpha} - e^{-\pi i \alpha}) \left(e^{\pi i \alpha} \int_{L_A^3} + \int_{L_A^4} \right) f(v) dv, \end{aligned}$$

where the principal branch is used for all complex powers in all integrals. Since the integral of f converges at $v = 0$, we can take $\epsilon \rightarrow 0$ in the first term on the right-hand side, which gives

$$F(w_1, w_2) = (1 - e^{-\pi i \alpha})(e^{2\pi i \alpha} - 1) \int_0^A f(v) dv + (e^{\pi i \alpha} - e^{-\pi i \alpha}) \left(e^{\pi i \alpha} \int_{L_A^3} + \int_{L_A^4} \right) f(v) dv. \quad (9.12)$$

As $\alpha \rightarrow n$, we have

$$\begin{aligned} (1 - e^{-\pi i \alpha})(e^{2\pi i \alpha} - 1) &= 2i\pi(-1)^n((-1)^n - 1)(\alpha - n) - 2\pi^2(\alpha - n)^2 + O((\alpha - n)^3), \\ e^{\pi i \alpha} - e^{-\pi i \alpha} &= 2i(-1)^n\pi(\alpha - n) + O((\alpha - n)^3), \\ (e^{\pi i \alpha} - e^{-\pi i \alpha})e^{\pi i \alpha} &= 2i\pi(\alpha - n) - 2\pi^2(\alpha - n)^2 + O((\alpha - n)^3), \end{aligned}$$

and

$$f(v) = e^{(\alpha-1)\ln v} e^{(\alpha-1)\ln(v-w_1)} e^{-\frac{\alpha}{2}\ln(v-w_2)} e^{-\frac{\alpha}{2}\ln(1-v)} = f_0 + (\alpha - n)f_1 + O((\alpha - n)^2).$$

Substituting these expansions into (9.12), we obtain the expansion (9.11) with F_1 and F_2 given for $n \geq 2$ by

$$F_1 = 2\pi i(1 - (-1)^n) \int_0^A f_0 dv + 2\pi i \left(\int_{L_A^3} + (-1)^n \int_{L_A^4} \right) f_0 dv,$$

and

$$F_2 = 2\pi i(1 - (-1)^n) \int_0^A f_1 dv - 2\pi^2 \left(\int_0^A + \int_{L_A^3} \right) f_0 dv + 2\pi i \left(\int_{L_A^3} + (-1)^n \int_{L_A^4} \right) f_1 dv.$$

The expression (9.9) for F_1 follows immediately if n is odd. If n is even, then f_0 has a pole of order $n/2$ at $v = 1$. Thus, choosing $A = 1 - \epsilon$ and using the residue theorem, we find

$$\begin{aligned} F_1(w_1, w_2) &= 2\pi i \int_{|v-1|=\epsilon} f_0 dv \\ &= (2\pi i)^2 \text{Res}_{v=1} \frac{v^{n-1}(v-w_1)^{n-1}(v-w_2)^{-\frac{n}{2}}}{(1-v)^{\frac{n}{2}}}, \quad n = 2, 4, \dots, \end{aligned} \quad (9.13)$$

which yields the expression (9.9) for F_1 also for even n . Finally, letting $A = 0$, we find the expression (9.10) for F_2 for n even. This completes the proof of (9.11).

We next claim that, as $\alpha \rightarrow n$,

$$\begin{aligned} &\text{Im} [\sigma(\theta^2)(-e^{i\theta^2})^{\alpha-1} F(\theta^1, \theta^2; \alpha)] \\ &= \begin{cases} -\text{Im} [e^{(n-1)i\theta^2} (F_2 + i(\theta^2 - 2\pi 1_{\{\theta^2 \geq \frac{\pi}{2}\}}) F_1)] (\alpha - n)^2 + O((\alpha - n)^3), & n = 2, 4, \dots, \\ -\text{Im} [e^{(n-1)i\theta^2} F_1] (\alpha - n) + O((\alpha - n)^2), & n = 3, 5, \dots \end{cases} \end{aligned} \quad (9.14)$$

Indeed, the expansion (9.11) yields

$$\begin{aligned}
\operatorname{Im} [\sigma(\theta^2)(-e^{i\theta^2})^{\alpha-1}F(\theta^1, \theta^2; \alpha)] &= -\operatorname{Im} \left[(1 \mp i\pi(\alpha - n) + \dots)e^{(n-1)i\theta^2} \right. \\
&\quad \times (1 + \ln(-e^{i\theta^2})(\alpha - n) + \dots)(F_1(\alpha - n) + F_2(\alpha - n)^2 + \dots) \Big] \\
&= -\operatorname{Im} [e^{(n-1)i\theta^2}F_1](\alpha - n) - \operatorname{Im} [e^{(n-1)i\theta^2}(F_2 \mp i\pi F_1 + i(\theta^2 - \pi)F_1)](\alpha - n)^2 \\
&\quad + O((\alpha - n)^3).
\end{aligned}$$

where the upper (lower) sign applies for $\theta^2 \geq \pi/2$ ($\theta^2 < \pi/2$) and we have used that $\ln(-e^{i\theta^2}) = i(\theta^2 - \pi)$ in the last step. Equation (9.14) therefore follows if we can show that

$$\operatorname{Im} [e^{(n-1)i\theta^2}F_1] = 0, \quad n = 2, 4, 6, \dots \quad (9.15)$$

Let $n \geq 2$ be even and define $g(w)$ by

$$g(w) = (1 + e^{-i\theta^2}w)^{n-1}(1 + e^{i\theta^2}w)^{n-1}(w + \cos \theta^2 - \cot \theta^1 \sin \theta^2)^{-\frac{n}{2}}w^{-\frac{n}{2}}.$$

Then, by (9.13),

$$\begin{aligned}
\operatorname{Im} [e^{(n-1)i\theta^2}F_1] &= \operatorname{Im} \left[e^{(n-1)i\theta^2}2\pi i \int_{|v-1|=\epsilon} f_0 dv \right] \\
&= \operatorname{Im} \left[e^{(n-2)i\theta^2}2\pi i \int_{|w|=\epsilon} (1 + e^{-i\theta^2}w)^{n-1}(e^{-i\theta^2}w + e^{-2i\theta^2})^{n-1} \right. \\
&\quad \times (e^{-i\theta^2}(w + \cos \theta^2 - \cot \theta^1 \sin \theta^2))^{-\frac{n}{2}}(-e^{-i\theta^2}w)^{-\frac{n}{2}}dw \Big] \\
&= (-1)^{-n/2} \operatorname{Im} \left[2\pi i \int_{|w-1|=\epsilon} g(w)dw \right]
\end{aligned}$$

where we have used the change of variables $v = 1 + e^{-i\theta^2}w$ and the definitions (9.3) of w_1 and w_2 in second equality. Since $g(w) = \overline{g(\bar{w})}$, the identity

$$\int_{\gamma} \overline{g(w)}dw = \int_{\bar{\gamma}} \overline{g(\bar{v})}dv \quad (9.16)$$

valid for a general contour $\gamma \subset \mathbb{C}$, implies that $\int_{|w-1|=\epsilon} g(w)dw$ is pure imaginary. This proves (9.15) and hence also (9.14).

For each integer $n \geq 2$, we have the following asymptotic behavior of \hat{c}^{-1} as $\alpha \rightarrow n$:

$$\frac{1}{\hat{c}} = \begin{cases} -\frac{h_n}{(\alpha-n)^2} + O(\frac{1}{\alpha-n}), & n \text{ even}, \\ -\frac{h_n}{\alpha-n} + O(1), & n \text{ odd}, \end{cases} \quad n = 2, 3, 4, \dots \quad (9.17)$$

Substituting (9.14) and (9.17) into (9.2), we find (9.7). \square

By taking the limit as ϵ approaches zero in the integrals in (9.9) and (9.10), it is possible to derive explicit expressions for F_1 and F_2 , and hence also for the function h . We illustrate this by computing the $\text{SLE}_{\kappa}(2)$ Green's function explicitly for $\kappa = 4$, $\kappa = 8/3$, and $\kappa = 2$.

Proposition 9.5. For $\kappa = 4$, $\kappa = 8/3$, and $\kappa = 2$ (i.e. for $\alpha = 2, 3, 4$), the $SLE_\kappa(2)$ Green's function is given by equation (2.12) where $h(\theta^1, \theta^2)$ is given explicitly by

$$h(\theta^1, \theta^2) = \frac{1}{4\pi \sin(\theta^1 - \theta^2)} \{ \sin(2\theta^1 - 2\theta^2) + 2\theta^1(1 - \cos 2\theta^2) + 2\theta^2(\cos 2\theta^1 - 1) - \sin 2\theta^1 + \sin 2\theta^2 \}, \quad \kappa = 4, \quad (9.18)$$

$$\begin{aligned} h(\theta^1, \theta^2) = \frac{1}{30\pi(\cos(\theta^1 - \theta^2) + 1)} & \left\{ \sqrt{\sin \theta^1 \sin \theta^2} \left[-6 \cos \left(\frac{\theta^1 - 3\theta^2}{2} \right) \right. \right. \\ & + \cos \left(\frac{3\theta^1 - 5\theta^2}{2} \right) + \cos \left(\frac{5\theta^1 - 3\theta^2}{2} \right) - 6 \cos \left(\frac{3\theta^1 - \theta^2}{2} \right) - 38 \cos \left(\frac{\theta^1 + \theta^2}{2} \right) \\ & + 20 \cos \left(\frac{3\theta^1 + 3\theta^2}{2} \right) + 14 \cos \left(\frac{5\theta^1 + \theta^2}{2} \right) + 14 \cos \left(\frac{\theta^1 + 5\theta^2}{2} \right) \Big] \\ & - 2 \cos^2 \left(\frac{\theta^1 - \theta^2}{2} \right) \left[-9 \sin 2\theta^1 \sin 2\theta^2 + (7 \cos 2\theta^2 + 8) \cos 2\theta^1 + 8 \cos 2\theta^2 \right. \\ & \left. \left. - 23 \right] \arg \left(\cos \left(\frac{\theta^1 + \theta^2}{2} \right) + i \sqrt{\sin \theta^1 \sin \theta^2} \right) \right\}, \quad \kappa = 8/3, \end{aligned} \quad (9.19)$$

and

$$\begin{aligned} h(\theta^1, \theta^2) = \frac{1}{192\pi} & \left\{ \frac{72 \sin^5(\theta^1) \cos(\theta^2) \cos(\theta^1 - 3\theta^2)}{\sin^3(\theta^1 - \theta^2)} + \frac{\sin^3(\theta^2)}{(\cot \theta^1 - \cot \theta^2)^3} \right. \\ & \times \left[96 \frac{(3\theta^1 \cot \theta^2 + 2) \cot \theta^1 + \theta^1(3 - 2 \csc^2 \theta^1) - 3 \cot \theta^2}{\sin \theta^1} \right. \\ & + \csc^6(\theta^2) [3(16\theta^2(3 \sin(\theta^1 - 2\theta^2) + \sin \theta^1) \sin \theta^1 + 5 \sin 2\theta^2 - 4 \sin 4\theta^2) \sin \theta^1 \\ & + 6 \cos(\theta^1 - 6\theta^2) - \cos(3\theta^1 - 6\theta^2) + (75 \cos 2\theta^2 - 30 \cos 4\theta^2 - 33) \cos \theta^1 \\ & \left. \left. - 17 \cos 3\theta^1 \right] \right\}, \quad \kappa = 2. \end{aligned} \quad (9.20)$$

Proof. We give the proof for $\kappa = 4$. The proofs for $\kappa = 8/3$ and $\kappa = 2$ are similar. Let $n = 2$. As ϵ goes to zero, we have

$$\begin{aligned} \int_{L_0^3(\epsilon)} f_0 dv &= \int_{L_0^3(\epsilon)} \frac{v(v - w_1)}{(v - w_2)(1 - v)} dv \\ &= \frac{w_2(w_1 - w_2) \ln(v - w_2) - (w_1 - 1) \ln(1 - v) + v(1 - w_2)}{w_2 - 1} \Big|_{v=0}^{1+\epsilon-i0} \\ &= \frac{-(w_1 - 1) \ln \epsilon}{w_2 - 1} + J_1(w_1, w_2) + O(\epsilon), \end{aligned}$$

where the order one term $J(w_1, w_2)$ is given by

$$J_1(w_1, w_2) = \frac{w_2(w_1 - w_2)(\ln(1 - w_2) - \ln(-w_2)) - i\pi(w_1 - 1) - w_2 + 1}{w_2 - 1}.$$

On the other hand, since the function

$$\ln v + \ln(v - w_1) - \frac{\ln(v - w_2)}{2}$$

is analytic at $v = 1$, the residue theorem gives

$$\begin{aligned} \int_{|v-1|=\epsilon} f_1 dv &= \int_{|v-1|=\epsilon} \frac{v(v-w_1)}{(v-w_2)(1-v)} \left(\ln v + \ln(v-w_1) - \frac{\ln(v-w_2)}{2} - \frac{\ln(1-v)}{2} \right) dv \\ &= -2\pi i \frac{1-w_1}{1-w_2} \left(\ln 1 + \ln(1-w_1) - \frac{\ln(1-w_2)}{2} \right) \\ &\quad - \frac{1}{2} \int_0^{2\pi} \frac{(1+\epsilon e^{i\varphi})(1+\epsilon e^{i\varphi}-w_1)}{(1+\epsilon e^{i\varphi}-w_2)(-\epsilon e^{i\varphi})} \ln(-\epsilon e^{i\varphi}) i \epsilon e^{i\varphi} d\varphi \\ &= -2\pi i \frac{1-w_1}{1-w_2} \left(\ln(1-w_1) - \frac{\ln(1-w_2)}{2} \right) \\ &\quad + \frac{i}{2} \int_0^{2\pi} \frac{1-w_1}{1-w_2} (\ln \epsilon + i(\varphi - \pi)) d\varphi + O(\epsilon \ln \epsilon) \\ &= i\pi \frac{1-w_1}{1-w_2} \ln \epsilon + J_2(w_1, w_2) + O(\epsilon \ln \epsilon), \end{aligned}$$

where the order one term $J_2(w_1, w_2)$ is given by

$$J_2(w_1, w_2) = -2\pi i \frac{1-w_1}{1-w_2} \left(\ln(1-w_1) - \frac{\ln(1-w_2)}{2} \right).$$

Hence, since the singular terms of $O(\ln \epsilon)$ cancel,

$$\begin{aligned} F_2 &= 2\pi i \lim_{\epsilon \rightarrow 0} \left(\pi i \int_{L_0^3} f_0(v) dv + \int_{|v-1|=\epsilon} f_1(v) dv \right) \\ &= 2\pi i (\pi i J_1(w_1, w_2) + J_2(w_1, w_2)), \quad n = 2. \end{aligned} \quad (9.21)$$

On the other hand,

$$F_1 = (2\pi i)^2 \operatorname{Res}_{v=1} \frac{v(v-w_1)}{(v-w_2)(1-v)} = -(2\pi i)^2 \frac{1-w_1}{1-w_2} = \frac{4\pi^2 e^{-i\theta^2} \sin \theta^1}{\sin(\theta^1 - \theta^2)}, \quad n = 2. \quad (9.22)$$

The terms J_1 and J_2 involve the logarithms $\ln(1-w_1)$, $\ln(1-w_2)$, and $\ln(-w_2)$. The expressions (9.3) for w_1 and w_2 imply that (recall that principal branches are used for all logarithms)

$$\begin{aligned} \ln(1-w_1) &= \ln(e^{-2i\theta^2}) = 2i(\pi 1_{\{\theta^2 \geq \frac{\pi}{2}\}} - \theta^2), \\ \ln(1-w_2) &= \ln\left(-e^{-i\theta^2} \frac{\sin(\theta^2 - \theta^1)}{\sin \theta^1}\right) = \ln\left|\frac{\sin(\theta^2 - \theta^1)}{\sin \theta^1}\right| + i(\pi - \theta^2), \\ \ln(-w_2) &= \ln\left(-e^{-i(\theta^2 - \theta^1)} \frac{\sin \theta^2}{\sin \theta^1}\right) = \ln\left|\frac{\sin \theta^2}{\sin \theta^1}\right| + i(\pi + \theta^1 - \theta^2), \end{aligned} \quad (9.23)$$

for all $(\theta^1, \theta^2) \in \Delta$. For $n = 2$, equation (9.7) gives

$$h(\theta^1, \theta^2; 2) = \frac{\sin \theta^1}{2\pi^3} \operatorname{Im} \left[e^{i\theta^2} (F_2 + i(\theta^2 - 2\pi 1_{\{\theta^2 \geq \frac{\pi}{2}\}}) F_1) \right].$$

Substituting the expressions (9.21) and (9.22) into this formula and using (9.23), equation (9.18) follows after simplification. \square

10 Asymptotics of Coulomb gas integrals

The goal of this section is to suggest a methodology for rigorously establishing (at least some) important observables for SLE with multiple curves and/or insertions. As explained in Section 4, candidates for these observables can be obtained by applying a screening procedure to the local martingales of the form (4.2) obtained from conformal field theory. A key step in the analysis consists of choosing the appropriate screening integration contour. This contour is determined by the requirement that the associated local martingale satisfies the correct boundary conditions. For the Schramm observable for $\operatorname{SLE}_\kappa(2)$, it will be directly verified in Appendix A that the function $P(z, \xi)$ defined in (2.5) fulfills the correct boundary conditions. (This can also be proved using the method developed in this section.) For the Green's function, the analogous verification is more complicated and involves the asymptotic analysis of the integral (2.7) as two or more of the points z, \bar{z}, ξ^1, ξ^2 come together. In this section, we present an approach for computing asymptotics of this type to all orders. Subsequently, in Appendix B, we use this approach to verify that the function (2.8) satisfies the boundary conditions expected from the $\operatorname{SLE}_\kappa(2)$ Green's function.

10.1 Screening integrals

In the case of one screening integral, the screening procedure (see Section 4) gives rise to local SLE martingales involving expressions of the form

$$\int_A^{(z_1+, z_2+, z_1-, z_2-)} \prod_{j=1}^N (u - z_j)^{a_j} du, \quad (10.1)$$

where $\{z_j\}_1^N \subset \mathbb{C}$ is a finite collection of points, $\{a_j\}_1^N \subset \mathbb{R}$ is a set of exponents, and the Pochhammer integration contour encloses two of these points, say z_1 and z_2 . In order to find the boundary behavior of the local martingale, it is necessary to compute the asymptotics of the integral (10.1) as two or more of the z_j merge. This presents some difficulties, because if a point z_k , $k = 3, \dots, N$, approaches z_1 or z_2 , then the contour gets squeezed between the two points which, in general, gives rise to singular behavior.

By applying a linear fractional transformation, we may assume that $z_1 = 0$ and $z_2 = 1$ in (10.1); this yields an expression of the form

$$\int_A^{(0+, 1+, 0-, 1-)} v^{a_1} (1 - v)^{a_2} \prod_{j=3}^N (v - w_j)^{a_j} dv. \quad (10.2)$$

For $N = 3$, this expression defines a hypergeometric function and the asymptotics can be obtained from well-known identities and expansions of these functions. However, for $N \geq 4$, the analysis is more intricate and we have not been able to find a reference dealing with this case. In this section, we consider the class of integrals corresponding to $N = 4$. More precisely, we consider asymptotics of the function $F(w_1, w_2) := F(a, b, c, d; w_1, w_2)$ defined by

$$F(w_1, w_2) = \int_A^{(0+, 1+, 0-, 1-)} v^a (v - w_1)^b (v - w_2)^c (1 - v)^d dv, \quad (w_1, w_2) \in \mathcal{D}_0 \subset \mathbb{C}^2, \quad (10.3)$$

where $A \in (0, 1)$ is a base point, $a, b, c, d \in \mathbb{R}$ are real exponents, and w_1, w_2 are assumed to lie outside the contour. In order to make F single-valued, we have restricted the domain of definition in (10.3) to the domain $\mathcal{D}_0 \subset \mathbb{C}^2$ defined by

$$\mathcal{D}_0 = \{(w_1, w_2) \in \mathbb{C}^2 \mid w_2 \in \mathbb{C} \setminus [0, \infty), w_1 \in \mathbb{C} \setminus ([0, \infty) \cup \gamma_{(w_2, \infty)})\},$$

where $\gamma_{(w_2, \infty)} \subset \mathbb{C}$ denotes a branch cut from w_2 to ∞ (to be specific, we henceforth choose $\gamma_{(w, \infty)} = \{rw \mid r \geq 1\}$). The function F defined in (9.1) is the special case of (10.3) when

$$a = b = \alpha - 1, \quad c = d = -\frac{\alpha}{2}. \quad (10.4)$$

It will be clear from the analysis that a similar approach can be used also for $N \geq 5$.

We assume that $a, d \in \mathbb{R} \setminus \mathbb{Z}$ are not integers, because otherwise F vanishes identically. We note that F can be analytically continued to a multiple-valued analytic function of $w_1 \in \mathbb{C} \setminus \{0, 1, w_2\}$ and $w_2 \in \mathbb{C} \setminus \{0, 1\}$. We will compute the asymptotic behavior of F to all orders as one or both of the points w_1 and w_2 approach 0 or 1. The basic idea is the following: If we want to consider the limit $w_1 \rightarrow 0$ say, then we rewrite F as a sum of two terms. One term which is defined by the same integral as F except that w_1 is now assumed to lie inside the contour in the same component as 0, and a second term which is defined by a similar expression but with the Pochhammer contour enclosing $\{0, w_1\}$ instead of $\{0, 1\}$, see equation (10.24). The asymptotics of both of these terms can easily be computed to all orders by replacing $(v - w_1)^b$ by its asymptotic expansion as $w_1 \rightarrow 0$:

$$(v - w_1)^b \sim \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(b+1)}{\Gamma(b+1-k)} v^{b-k} \frac{w_1^k}{k!}. \quad (10.5)$$

We emphasize that we cannot, in general, compute the asymptotics of F as $w_1 \rightarrow 0$ by substituting the expansion (10.5) directly into (10.3). Indeed, such a procedure gives the correct contribution from the first term, but completely ignores the contribution from the second term.

10.2 The hypergeometric case of $N = 3$

Before turning to the case $N = 4$, let us consider, as motivation, the case $N = 3$ in which the integral in (10.2) reduces to a hypergeometric function.

Let $F(w)$ denote the expression in (10.2) for $N = 3$:

$$F(w) = \int_A^{(0+,1+,0-,1-)} v^a (v-w)^c (1-v)^d dv, \quad w \in \mathbb{C} \setminus [0, \infty). \quad (10.6)$$

If \tilde{F} denotes the function

$$\tilde{F}(w) = \int_A^{(0+,1+,0-,1-)} v^a (w-v)^c (1-v)^d dv, \quad w \in \mathbb{C} \setminus (-\infty, 1],$$

where w lies exterior to the contour, then the definition (8.13) of ${}_2F_1$ implies

$$\tilde{F}(w) = \frac{4\pi^2 w^c {}_2F_1(-c, a+1, a+d+2; 1/w)}{e^{-(a+d+2)\pi i} \Gamma(a+d+2) \Gamma(-a) \Gamma(-d)}, \quad w \in \mathbb{C} \setminus (-\infty, 1]. \quad (10.7)$$

On the other hand, F and \tilde{F} are related by

$$F(w) = \rho(w) \tilde{F}(w), \quad w_1, w_2 \in \mathbb{C} \setminus \mathbb{R},$$

where

$$\rho(w) = \begin{cases} e^{-i\pi c}, & \text{Im } w > 0, \\ e^{i\pi c}, & \text{Im } w < 0. \end{cases} \quad (10.8)$$

Thus we can use the relation (10.7) to derive asymptotic expansions of $F(w)$ as $w \rightarrow 0$ and $w \rightarrow 1$. For definiteness, let us consider the limit $w \rightarrow 1$. The function ${}_2F_1$ is an analytic function of z with a branch cut along $[1, \infty)$; in particular, it is not analytic at $z = 1$. In order to arrive at the correct asymptotics as $w \rightarrow 1$, we therefore first use the hypergeometric identity (Eq. (10.12) in [34])

$$\begin{aligned} {}_2F_1(a, b, c; z) &= \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} {}_2F_1(a, b, 1+a+b-c; 1-z) \\ &\quad + \frac{\Gamma(c) \Gamma(a+b-c)}{\Gamma(a) \Gamma(b)} (1-z)^{c-a-b} {}_2F_1(c-a, c-b, 1+c-a-b; 1-z), \\ &\quad z \in \mathbb{C} \setminus ((-\infty, 0] \cup [1, \infty)), \end{aligned} \quad (10.9)$$

to rewrite equation (10.7) as

$$\begin{aligned} \tilde{F}(w) &= - \frac{4\pi e^{i\pi(a+d+2)} \sin(\pi a) \Gamma(a+1) w^c \sin(\pi d) \csc(\pi(c+d))}{\Gamma(-c-d) \Gamma(a+c+d+2)} \\ &\quad \times {}_2F_1\left(1+a, -c, -c-d; 1-\frac{1}{w}\right) \\ &\quad + \frac{4\pi e^{i\pi(a+d+2)} \sin(\pi a) \sin(\pi d) \Gamma(d+1)}{\Gamma(-c) \Gamma(c+d+2) \sin(\pi(c+d))} w^c \left(\frac{w-1}{w}\right)^{c+d+1} \\ &\quad \times {}_2F_1\left(d+1, a+c+d+2; c+d+2; 1-\frac{1}{w}\right), \quad w \in \mathbb{C} \setminus (-\infty, 1]. \end{aligned} \quad (10.10)$$

The hypergeometric functions in (10.10) are analytic at $w = 1$. In fact,

$${}_2F_1(a, b, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

where the Pochhammer symbol $(a)_k$ is defined by

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = (a+k-1) \cdots (a+2)(a+1)a.$$

Using this expansion in (10.10), we find the following representation which gives the asymptotics of $\tilde{F}(w)$ as $w \rightarrow 1$ to all orders:

$$\tilde{F}(w) = P_1(w) + (w-1)^{c+d+1} P_2(w) \quad (10.11)$$

where

$$P_1(w) = (-1 + e^{2\pi i a} - e^{2\pi i(a+d)} + e^{2\pi i d}) \hat{P}_1(w), \quad (10.12a)$$

$$P_2(w) = (-1 + e^{2\pi i a} - e^{2\pi i(a+d)} + e^{2\pi i d}) \hat{P}_2(w), \quad (10.12b)$$

and

$$\begin{aligned} \hat{P}_1(w) &= \sum_{k=0}^{\infty} \frac{\pi \Gamma(a+1) \Gamma(c+1) \csc((c+d+1-k)\pi)}{\Gamma(c+1-k) \Gamma(k-c-d) \Gamma(2+a+c+d-k)} \frac{(w-1)^k}{k!}, \\ \hat{P}_2(w) &= \sum_{k=0}^{\infty} \frac{\pi \Gamma(a+1) \Gamma(d+1+k) \csc((c+d)\pi)}{\Gamma(a+1-k) \Gamma(-c) \Gamma(c+d+2+k)} \frac{(w-1)^k}{k!}. \end{aligned}$$

10.3 Asymptotics of F as $w_2 \rightarrow 1$

Let us now consider the case of $N = 4$ in which $F(w_1, w_2)$ is given by (10.3). We first consider the asymptotics of F as $w_2 \rightarrow 1$. The basic idea is to derive generalizations of the hypergeometric identities (10.9) and (10.11).

Let $\mathcal{D}_1 \subset \mathbb{C}^2$ denote the domain

$$\mathcal{D}_1 = \{(w_1, w_2) \in \mathbb{C}^2 \mid w_1 \in \mathbb{C} \setminus [0, \infty), w_2 \in \mathbb{C} \setminus ((-\infty, 1] \cup \gamma_{(w_1, \infty)})\}, \quad (10.13)$$

and define \tilde{F} by

$$\tilde{F}(w_1, w_2) = \int_A^{(0+, 1+, 0-, 1-)} v^a (v-w_1)^b (w_2-v)^c (1-v)^d dv, \quad (w_1, w_2) \in \mathcal{D}_1, \quad (10.14)$$

where w_1, w_2 lie exterior to the contour. Then

$$F(w_1, w_2) = \rho(w_2) \tilde{F}(w_1, w_2), \quad (w_1, w_2) \in \mathcal{D}_0 \cap \mathcal{D}_1, \quad (10.15)$$

where ρ is the function in (10.8).

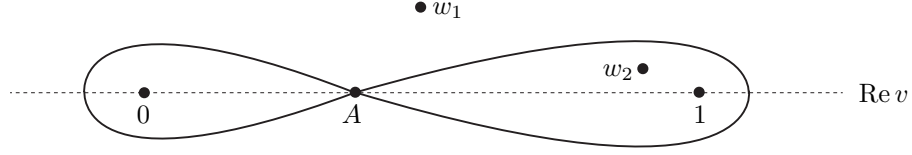


Figure 9. In the definition of the function P_1 , the point w_1 lies exterior to the contour, whereas w_2 lies inside the contour in the same component as 1.

Assuming that $c + d \notin \mathbb{Z}$, we define two functions $P_j : \mathcal{D}_1 \rightarrow \mathbb{C}$, $j = 1, 2$, as follows. The function P_1 is defined (up to a constant) by the same formula as \tilde{F} except that the point w_2 is assumed to lie inside the contour in the same component as 1; more precisely,

$$P_1(w_1, w_2) = \frac{e^{-i\pi c} \sin(d\pi)}{\sin(\pi(d+c))} \int_A^{(0+, 1+, 0-, 1-)} v^a (v - w_1)^b (w_2 - v)^c (1 - v)^d dv, \quad (w_1, w_2) \in \mathcal{D}_1,$$

where w_1 lies outside the contour and w_2 lies inside the contour in the same component as 1, see Figure 9. The function $P_2 : \mathcal{D}_1 \rightarrow \mathbb{C}$ is defined as follows. First, given $w_1 \in \mathbb{C} \setminus [0, \infty)$, we define $P_2(w_1, w_2)$ for $\operatorname{Re} w_2 \in (0, 1)$ with $\operatorname{Im} w_2 > 0$ sufficiently small by

$$P_2(w_1, w_2) = \frac{e^{i\pi(a+d)} \sin(a\pi)}{\sin(\pi(d+c))} e^{-i\pi(c+d+1)} \times \int_A^{(0+, 1+, 0-, 1-)} (w_2 + s(1 - w_2))^a (w_2 - w_1 + s(1 - w_2))^b s^c (1 - s)^d ds, \quad (10.16)$$

where $A \in (0, 1)$ and the points $\frac{w_2}{w_2 - 1}$ and $\frac{w_1 - w_2}{1 - w_2}$ are assumed to lie exterior to the contour. Then, for each $w_1 \in \mathbb{C} \setminus [0, \infty)$, we use analytic continuation to extend P_2 to a (single-valued) analytic function of $w_2 \in \mathbb{C} \setminus ((-\infty, 1] \cup \gamma_{(w_1, \infty)})$. The latter step is permissible because the function P_2 can be analytically continued as long as the points $\frac{w_2}{w_2 - 1}$ and $\frac{w_1 - w_2}{1 - w_2}$ stay away from the set $\{0, 1, \infty\}$, i.e., as long as $w_2 \notin \{0, 1, w_1, \infty\}$.

Let $f(w_1, w_2 \pm i0)$ denote the boundary values of a function $f(w_1, w_2)$ as w_2 approaches the real axis from above and below, respectively.

Lemma 10.1. *Suppose $a, d, c + d \notin \mathbb{Z}$. Then*

$$\tilde{F}(w_1, w_2) = P_1(w_1, w_2) + (w_2 - 1)^{c+d+1} P_2(w_1, w_2), \quad (w_1, w_2) \in \mathcal{D}_1. \quad (10.17)$$

Proof. It is enough to show that

$$\begin{aligned} \tilde{F}(w_1, w_2 + i0) &= P_1(w_1, w_2 + i0) + |1 - w_2|^{c+d+1} e^{i\pi(c+d+1)} P_2(w_1, w_2 + i0), \\ &\quad w_1 \in \mathbb{C} \setminus [0, \infty), \quad w_2 \in (0, 1). \end{aligned} \quad (10.18)$$

Indeed, for each $w_1 \in \mathbb{C} \setminus [0, \infty)$, both sides of the equation (10.17) are analytic functions of $w_2 \in \mathbb{C} \setminus ((-\infty, 1] \cup \gamma_{(w_1, \infty)})$ which can be extended to multiple-valued analytic functions of $w_2 \in \mathbb{C} \setminus \{0, 1, w_1\}$. Hence (10.17) follows from (10.18) by analytic continuation.

Let us prove (10.18). Let $\epsilon > 0$ be small. Let $w_1 \in \mathbb{C} \setminus [0, \infty)$ and $w_2 \in (0, 1)$. Given $w \in \mathbb{C}$, let

$$S_w^+ = \{w + \epsilon e^{i\phi} \mid 0 \leq \phi \leq \pi\}, \quad S_w^- = \{w + \epsilon e^{i\phi} \mid -\pi \leq \phi \leq 0\},$$

denote counterclockwise semicircles of radius ϵ centered at w . Here and below we adopt the convention that unless the integration contour is a Pochhammer contour, the principal branch is used for all complex powers and logarithms in the integrand. Then we can write

$$\begin{aligned} \tilde{F}(w_1, w_2 + i0) = & \left\{ \int_{L_{w_2-\epsilon}^1} + e^{2\pi ia} \int_{L_{w_2-\epsilon}^2 + S_{w_2}^- + L_{w_2+\epsilon}^3} + e^{2\pi i(a+d+c)} \int_{L_{w_2+\epsilon}^4} \right. \\ & - e^{2\pi i(a+d)} \int_{S_{w_2}^- + L_{w_2-\epsilon}^2} + e^{2\pi id} \int_{-L_{w_2-\epsilon}^1 + S_{w_2}^-} - e^{2\pi i(d+c)} \int_{L_{w_2+\epsilon}^4} \\ & \left. - \int_{L_{w_2+\epsilon}^3 + S_{w_2}^-} \right\} v^a (v - w_1)^b (w_2 - v)^c (1 - v)^d dv \end{aligned}$$

and

$$\begin{aligned} P_1(w_1, w_2 + i0) = & \frac{e^{-i\pi c} \sin(d\pi)}{\sin((c+d)\pi)} \left\{ \int_{L_{w_2-\epsilon}^1} + e^{2\pi ia} \int_{L_{w_2-\epsilon}^2 + S_{w_2}^- + L_{w_2+\epsilon}^3} + e^{2\pi i(a+d+c)} \int_{L_{w_2+\epsilon}^4 + S_{w_2}^+ - L_{w_2-\epsilon}^2} \right. \\ & \left. - e^{2\pi i(d+c)} \int_{L_{w_2-\epsilon}^1 + S_{w_2}^+ + L_{w_2+\epsilon}^4} - \int_{L_{w_2+\epsilon}^3 + S_{w_2}^-} \right\} v^a (v - w_1)^b (w_2 - v)^c (1 - v)^d dv. \end{aligned}$$

Simplification gives

$$\begin{aligned} \tilde{F}(w_1, w_2 + i0) = & \left\{ (1 - e^{2\pi id}) \left(\int_{L_{w_2-\epsilon}^1} + e^{2\pi ia} \int_{L_{w_2-\epsilon}^2} \right) \right. \\ & + (e^{2\pi ia} - 1) \left(\int_{L_{w_2+\epsilon}^3 + S_{w_2}^-} + e^{2\pi i(d+c)} \int_{L_{w_2+\epsilon}^4} - e^{2\pi id} \int_{S_{w_2}^-} \right) \left. \right\} \\ & \times v^a (v - w_1)^b (w_2 - v)^c (1 - v)^d dv \end{aligned}$$

and

$$\begin{aligned} P_1(w_1, w_2 + i0) = & \frac{e^{2i\pi d} - 1}{e^{2\pi i(d+c)} - 1} \left\{ (1 - e^{2\pi i(d+c)}) \left(\int_{L_{w_2-\epsilon}^1} + e^{2\pi ia} \int_{L_{w_2-\epsilon}^2} \right) \right. \\ & + (e^{2\pi ia} - 1) \left(\int_{L_{w_2+\epsilon}^3 + S_{w_2}^-} + e^{2\pi i(d+c)} \int_{L_{w_2+\epsilon}^4 + S_{w_2}^+} \right) \left. \right\} \\ & \times v^a (v - w_1)^b (w_2 - v)^c (1 - v)^d dv. \end{aligned}$$

Hence

$$\begin{aligned} \tilde{F}(w_1, w_2 + i0) - P_1(w_1, w_2 + i0) = & \frac{(e^{2\pi ia} - 1) \sin(c\pi) e^{i\pi d}}{\sin(\pi(d+c))} \left\{ \int_{L_{w_2+\epsilon}^3} + e^{2\pi i(d+c)} \int_{L_{w_2+\epsilon}^4} \right. \\ & \left. + \frac{e^{2\pi ic} (e^{2\pi id} - 1)}{1 - e^{2\pi ic}} \int_{S_{w_2}^+ + S_{w_2}^-} \right\} v^a (v - w_1)^b (w_2 - v)^c (1 - v)^d dv. \end{aligned}$$

Using the identity

$$(w_2 - v)^c = \begin{cases} e^{-i\pi c}(v - w_2)^c, & v \in S_{w_2}^+ \cup L_{w_2+\epsilon}^4, \\ e^{i\pi c}(v - w_2)^c, & v \in S_{w_2}^- \cup L_{w_2+\epsilon}^3, \end{cases}$$

we can write this as

$$\begin{aligned} \tilde{F}(w_1, w_2 + i0) - P_1(w_1, w_2 + i0) &= \frac{(e^{2\pi ia} - 1) \sin(c\pi) e^{i\pi(d+c)}}{\sin(\pi(d+c))} \left\{ \int_{L_{w_2+\epsilon}^3} + e^{2\pi id} \int_{L_{w_2+\epsilon}^4} \right. \\ &+ \left. \frac{e^{2\pi id} - 1}{1 - e^{2\pi ic}} \int_{S_{w_2}^+} + \frac{e^{2\pi ic}(e^{2\pi id} - 1)}{1 - e^{2\pi ic}} \int_{S_{w_2}^-} \right\} \\ &\times v^a(v - w_1)^b(v - w_2)^c(1 - v)^d dv, \quad w_1 \in \mathbb{C} \setminus [0, \infty), \quad w_2 \in (0, 1), \end{aligned}$$

Factoring out $\frac{1}{e^{2\pi ic} - 1}$, we obtain

$$\begin{aligned} \tilde{F}(w_1, w_2 + i0) - P_1(w_1, w_2 + i0) &= \frac{(e^{2\pi ia} - 1) \sin(c\pi) e^{i\pi(d+c)}}{\sin(\pi(d+c))(e^{2\pi ic} - 1)} \left\{ (e^{2\pi ic} - 1) \int_{L_{w_2+\epsilon}^3} \right. \\ &+ e^{2\pi id}(e^{2\pi ic} - 1) \int_{L_{w_2+\epsilon}^4} + (1 - e^{2\pi id}) \int_{S_{w_2}^+} + e^{2\pi ic}(1 - e^{2\pi id}) \int_{S_{w_2}^-} \left. \right\} \\ &\times v^a(v - w_1)^b(v - w_2)^c(1 - v)^d dv. \end{aligned}$$

That is,

$$\begin{aligned} \tilde{F}(w_1, w_2 + i0) - P_1(w_1, w_2 + i0) &= \frac{(e^{2\pi ia} - 1) \sin(c\pi) e^{i\pi(d+c)}}{\sin(\pi(d+c))(e^{2\pi ic} - 1)} \int_{w_2+\epsilon}^{(w_2+, 1+, w_2-, 1-)} \\ &\times v^a(v - w_1)^b(v - w_2)^c(1 - v)^d dv. \end{aligned}$$

Performing the change of variables $s = \frac{v-w_2}{1-w_2}$, which maps the interval $(w_2, 1)$ to the interval $(0, 1)$, this yields

$$\begin{aligned} \tilde{F}(w_1, w_2 + i0) - P_1(w_1, w_2 + i0) &= \frac{e^{i(a+d)\pi} \sin(a\pi)}{\sin(\pi(d+c))} \int_A^{(0+, 1+, 0-, 1-)} (w_2 + s(1 - w_2))^a \\ &\times (w_2 + s(1 - w_2) - w_1)^b (s(1 - w_2))^c ((1 - w_2)(1 - s))^d (1 - w_2) ds. \end{aligned}$$

Comparing this expression with the definition (10.16) of P_2 , equation (10.18) follows. \square

Using the identity (10.17), we can easily find the asymptotics of $F = \rho(w_2)\tilde{F}$ as $w_2 \rightarrow 1$ to all orders. Indeed, the functions P_1 and P_2 in (10.17) admit asymptotic expansions to all orders as follows. Substituting the expansion

$$\begin{aligned} (w_2 - v)^c &\sim (1 - v)^c + c(w_2 - 1)(1 - v)^{c-1} + \frac{c(c-1)(w_2 - 1)^2}{2}(1 - v)^{c-2} + \dots \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(c+1)}{\Gamma(c+1-k)} \frac{(w_2 - 1)^k}{k!} (1 - v)^{c-k}, \quad w_2 \rightarrow 1, \end{aligned}$$

into the definition of $P_1(w_1, w_2)$ and recalling that w_2 and 1 lie in the same component inside the contour, we find

$$P_1(w_1, w_2) \sim \frac{e^{-i\pi c} \sin(d\pi)}{\sin(\pi(d+c))} \sum_{k=0}^{\infty} \frac{\Gamma(c+1)}{\Gamma(c+1-k)} \frac{(w_2-1)^k}{k!} \\ \times \int_A^{(0+, 1+, 0-, 1-)} v^a (v-w_1)^b (1-v)^{d+c-k} dv, \quad w_2 \rightarrow 1, \quad (10.19)$$

where the integral on the right-hand side can be expressed in terms of hypergeometric functions if desired. Similarly, substituting the expansions

$$(w_2 + s(1-w_2))^a \sim \sum_{k=0}^{\infty} \frac{\Gamma(a+1)}{\Gamma(a+1-k)} (1-s)^k \frac{(w_2-1)^k}{k!}, \quad w_2 \rightarrow 1,$$

and

$$(w_2 + s(1-w_2) - w_1)^b \sim \sum_{l=0}^{\infty} \frac{\Gamma(a+1)}{\Gamma(a+1-l)} (1-s)^l (1-w_1)^{b-l} \frac{(w_2-1)^l}{l!}, \quad w_2 \rightarrow 1,$$

into the definition of $P_2(w_1, w_2)$, we find, as $w_2 \rightarrow 1$,

$$P_2(w_1, w_2) \sim \frac{e^{i\pi(a+d)} \sin(a\pi)}{\sin(\pi(d+c))} e^{-i\pi(c+d+1)} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{\Gamma(a+1)}{\Gamma(a+1-k)} \frac{\Gamma(a+1)}{\Gamma(a+1-l)} \\ \times \frac{(w_2-1)^k}{k!} (1-w_1)^{b-l} \frac{(w_2-1)^l}{l!} \int_A^{(0+, 1+, 0-, 1-)} s^c (1-s)^{d+k+l} ds. \quad (10.20)$$

10.4 Asymptotics of F as $w_1 \rightarrow 0$

In order to determine the asymptotics of $F(w_1, w_2)$ as $w_1 \rightarrow 0$, we define two functions $Q_j : \mathcal{D}_0 \rightarrow \mathbb{C}$, $j = 1, 2$, as follows. The function $Q_1(w_1, w_2)$ is defined by

$$Q_1(w_1, w_2) = \frac{e^{2\pi i a} - 1}{e^{2\pi i(a+b)} - 1} \int_A^{(0+, 1+, 0-, 1-)} v^a (v-w_1)^b (v-w_2)^c (1-v)^d dv, \quad (w_1, w_2) \in \mathcal{D}_0,$$

where $A \in (0, 1)$, w_1 lies inside the contour in the same component as 0, and w_2 lies outside the contour. Given $w_2 \in \mathbb{C} \setminus [0, \infty)$, we define $Q_2(w_1, w_2)$ for $\operatorname{Re} w_1 \in (0, 1)$ with $\operatorname{Im} w_1 < 0$ sufficiently small by

$$Q_2(w_1, w_2) = \frac{(e^{2\pi i d} - 1)e^{-i\pi b}}{1 - e^{-2i\pi(a+b)}} \int_A^{(0+, 1+, 0-, 1-)} s^a (1-s)^b (sw_1 - w_2)^c (1-sw_1)^d ds, \quad (10.21)$$

where $A \in (0, 1)$ and the points $\frac{w_2}{w_1}$ and $\frac{1}{w_1}$ lie exterior to the contour. For each $w_2 \in \mathbb{C} \setminus [0, \infty)$, we then use analytic continuation to extend Q_2 to a function of $w_1 \in \mathbb{C} \setminus ([0, \infty) \cup \gamma_{(w_2, \infty)})$.

Lemma 10.2. *Suppose $a, d, a+b \notin \mathbb{Z}$. Then*

$$F(w_1, w_2) = Q_1(w_1, w_2) + w_1^{a+b+1} Q_2(w_1, w_2), \quad (w_1, w_2) \in \mathcal{D}_0. \quad (10.22)$$

Proof. By analyticity, is enough to show that

$$F(w_1 - i0, w_2) = Q_1(w_1 - i0, w_2) + w_1^{a+b+1} Q_2(w_1 - i0, w_2) \quad (10.23)$$

for $w_1 \in (0, 1)$ and $w_2 \in \mathbb{C} \setminus [0, \infty)$.

Let $\epsilon > 0$ be small. Let $w_1 \in (0, 1)$ and $w_2 \in \mathbb{C} \setminus [0, \infty)$. Then

$$\begin{aligned} F(w_1 - i0, w_2) = & \left\{ \int_{L_{w_1-\epsilon}^1} + e^{2\pi i(a+b)} \int_{L_{w_1-\epsilon}^2} - e^{2\pi i a} \int_{S_{w_1}^+} + e^{2\pi i a} \int_{L_{w_1+\epsilon}^3} + e^{2\pi i(a+d)} \int_{L_{w_1+\epsilon}^4} \right. \\ & + e^{2\pi i(a+d)} \int_{S_{w_1}^+} - e^{2\pi i(a+b+d)} \int_{L_{w_1-\epsilon}^2} - e^{2\pi i d} \int_{L_{w_1-\epsilon}^1 + S_{w_1}^+ + L_{w_1+\epsilon}^4} \\ & \left. - \int_{L_{w_1+\epsilon}^3} + \int_{S_{w_1}^+} \right\} v^a (v - w_1)^b (v - w_2)^c (1 - v)^d dv, \end{aligned}$$

where w_2 lies exterior to the contours. Moreover,

$$\begin{aligned} Q_1(w_1 - i0, w_2) = & \frac{e^{2\pi i a} - 1}{e^{2\pi i(a+b)} - 1} \left\{ \int_{S_w^+ + L_{w_1-\epsilon}^1} + e^{2\pi i(a+b)} \int_{L_{w_1-\epsilon}^2 + S_w^- + L_{w+\epsilon}^3} \right. \\ & + e^{2\pi i(a+b+d)} \int_{L_{w+\epsilon}^4 - S_w^- - L_{w-\epsilon}^2} - e^{2\pi i d} \int_{L_{w-\epsilon}^1 + S_w^+ + L_{w+\epsilon}^4} + \int_{-L_{w+\epsilon}^3} \left. \right\} \\ & \times v^a (v - w_1)^b (v - w_2)^c (1 - v)^d dv. \end{aligned}$$

Simplification gives

$$\begin{aligned} F(w_1 - i0, w_2) = & \left\{ (1 - e^{2\pi i d}) \int_{L_{w_1-\epsilon}^1} + e^{2\pi i(a+b)} (1 - e^{2\pi i d}) \int_{L_{w_1-\epsilon}^2} \right. \\ & + (e^{2\pi i a} - 1) \left((e^{2\pi i d} - 1) \int_{S_{w_1}^+} + \int_{L_{w_1+\epsilon}^3} + e^{2\pi i d} \int_{L_{w_1+\epsilon}^4} \right) \left. \right\} \\ & \times v^a (v - w_1)^b (v - w_2)^c (1 - v)^d dv \end{aligned}$$

and

$$\begin{aligned} Q_1(w_1 - i0, w_2) = & \frac{e^{2\pi i a} - 1}{e^{2\pi i(a+b)} - 1} \left\{ (1 - e^{2\pi i d}) \int_{L_{w_1-\epsilon}^1} + e^{2\pi i(a+b)} (1 - e^{2\pi i d}) \int_{L_{w_1-\epsilon}^2} \right. \\ & + e^{2\pi i(a+b)} (1 - e^{2\pi i d}) \int_{S_w^-} + (e^{2\pi i(a+b)} - 1) \int_{L_{w+\epsilon}^3} \\ & + e^{2\pi i d} (e^{2\pi i(a+b)} - 1) \int_{L_{w+\epsilon}^4} + (1 - e^{2\pi i d}) \int_{S_w^+} \left. \right\} \\ & \times v^a (v - w_1)^b (v - w_2)^c (1 - v)^d dv. \end{aligned}$$

Hence

$$F(w_1 - i0, w_2) - Q_1(w_1 - i0, w_2) = \frac{(e^{2\pi i d} - 1)e^{-2i\pi b}}{1 - e^{-2i\pi(a+b)}} \left\{ (1 - e^{2i\pi b}) \int_{L_{w_1-\epsilon}^1} \right.$$

$$+ e^{2\pi i(a+b)}(1 - e^{2i\pi b}) \int_{L_{w_1-\epsilon}^2} + e^{2\pi i b}(e^{2\pi i a} - 1) \int_{S_{w_1}^+ + S_{w_1}^-} \Big\} v^a (v - w_1)^b (v - w_2)^c (1 - v)^d dv.$$

Using the identity

$$(v - w_1)^b = \begin{cases} e^{i\pi b}(w_1 - v)^b, & v \in S_w^+ \cup L_{w+\epsilon}^1, \\ e^{-i\pi b}(w_1 - v)^b, & v \in S_w^- \cup L_{w+\epsilon}^2, \end{cases}$$

we can write this as

$$\begin{aligned} F(w_1 - i0, w_2) - Q_1(w_1 - i0, w_2) &= \frac{(e^{2\pi i d} - 1)e^{-i\pi b}}{1 - e^{-2i\pi(a+b)}} \left\{ (1 - e^{2i\pi b}) \int_{L_{w_1-\epsilon}^1} \right. \\ &\quad \left. + e^{2\pi i a}(1 - e^{2i\pi b}) \int_{L_{w_1-\epsilon}^2} + e^{2\pi i b}(e^{2\pi i a} - 1) \int_{S_{w_1}^+} + (e^{2\pi i a} - 1) \int_{S_{w_1}^-} \right\} \\ &\quad \times v^a (w_1 - v)^b (v - w_2)^c (1 - v)^d dv, \quad w_1 \in (0, 1), \quad w_2 \in \mathbb{C} \setminus [0, \infty). \end{aligned}$$

That is,

$$\begin{aligned} F(w_1 - i0, w_2) - Q_1(w_1 - i0, w_2) &= \frac{(e^{2\pi i d} - 1)e^{-i\pi b}}{1 - e^{-2i\pi(a+b)}} \int_{w_1-\epsilon}^{(0+, w_1+, 0-, w_1-)} \\ &\quad \times v^a (w_1 - v)^b (v - w_2)^c (1 - v)^d dv. \end{aligned} \quad (10.24)$$

Performing the change of variables $s = \frac{v}{w_1}$, which maps the interval $(0, w_1)$ to the interval $(0, 1)$, we obtain

$$\begin{aligned} F(w_1 - i0, w_2) - Q_1(w_1 - i0, w_2) &= \frac{(e^{2\pi i d} - 1)e^{-i\pi b}}{1 - e^{-2i\pi(a+b)}} w_1^{a+b+1} \int_A^{(0+, 1+, 0-, 1-)} \\ &\quad \times s^a (1 - s)^b (sw_1 - w_2)^c (1 - sw_1)^d ds. \end{aligned}$$

The lemma follows. \square

In the same way that (10.17) gives the asymptotics as $w_2 \rightarrow 1$, the identity (10.22) gives an asymptotic expansion of $F(w_1, w_2)$ as $w_1 \rightarrow 0$ to all orders.

10.5 Asymptotics of F as $w_1 \rightarrow 0$ and $w_2 \rightarrow 0$

We next determine the asymptotics of $F(w_1, w_2)$ in the regime where both w_1 and w_2 approach zero. Assuming $a + b, a + b + c \notin \mathbb{Z}$, we define two functions $R_j : \mathcal{D}_0 \rightarrow \mathbb{C}$, $j = 1, 2$, as follows. The function $R_1(w_1, w_2)$ is defined for $(w_1, w_2) \in \mathcal{D}_0$ by

$$R_1(w_1, w_2) = \frac{e^{2\pi i a} - 1}{e^{2\pi i(a+b+c)} - 1} \int_A^{(0+, 1+, 0-, 1-)} v^a (v - w_1)^b (v - w_2)^c (1 - v)^d dv,$$

where $A \in (0, 1)$ and both points w_1 and w_2 are assumed to lie inside the contour in the same component as 0. For $0 < \operatorname{Re} w_1 < \operatorname{Re} w_2 < 1$ with $\operatorname{Im} w_1 < 0$ and $\operatorname{Im} w_2 < 0$, we define $R_2(w_1, w_2)$ by

$$R_2(w_1, w_2) = \frac{e^{2\pi i(a+b)}(e^{2\pi i a} - 1)(e^{2\pi i d} - 1)e^{i\pi c}}{(e^{2\pi i(a+b)} - 1)(e^{2\pi i(a+b+c)} - 1)}$$

$$\times \int_A^{(0+,1+,0-,1-)} s^a (sw_2 - w_1)^b (1-s)^c (1-sw_2)^d ds, \quad (10.25)$$

where we assume $A \in (0,1)$ is so large that $\operatorname{Re}(Aw_2 - w_1) > 0$, that the point $\frac{w_1}{w_2}$ lies inside the contour in the same component as 0, and that $\frac{1}{w_2}$ lies outside the contour. We then use analytic continuation to extend R_2 to all of \mathcal{D}_0 .

Lemma 10.3. *Suppose $a, d, a+b, a+b+c \notin \mathbb{Z}$. Then*

$$F(w_1, w_2) = R_1(w_1, w_2) + w_2^{a+c+1} R_2(w_1, w_2) + w_1^{a+b+1} Q_2(w_1, w_2), \quad (w_1, w_2) \in \mathcal{D}_0. \quad (10.26)$$

Proof. Both sides of (10.26) are analytic functions of $(w_1, w_2) \in \mathcal{D}_0$ which extend to multiple-valued analytic functions of

$$(w_1, w_2) \in \mathbb{C}^2 \setminus (\{w_1 = 0\} \cup \{w_1 = 1\} \cup \{w_2 = 0\} \cup \{w_2 = 1\} \cup \{w_1 = w_2\}). \quad (10.27)$$

Hence, by Lemma 10.2, it is enough to show that

$$Q_{1-}(w_1, w_2) = R_{1-}(w_1, w_2) + w_2^{a+c+1} R_{2-}(w_1, w_2), \quad 0 < w_1 < w_2 < 1, \quad (10.28)$$

where, for a function f , we use the short-hand notation $f_-(w_1, w_2) := f(w_1 - i0, w_2 - i0)$.

Let $0 < w_1 < w_2 < 1$ and suppose $0 < \epsilon < \frac{1}{2} \min\{w_1, w_2 - w_1, 1 - w_2\}$. Then

$$\begin{aligned} Q_{1-}(w_1, w_2) = & \frac{e^{2\pi ia} - 1}{e^{2\pi i(a+b)} - 1} \left\{ \int_{L_{w_2-\epsilon}^1} + e^{2\pi i(a+b+c)} \int_{L_{w_2-\epsilon}^2} - e^{2\pi i(a+b)} \int_{S_{w_2}^+} \right. \\ & + e^{2\pi i(a+b)} \int_{L_{w_2+\epsilon}^3} + e^{2\pi i(a+b+d)} \int_{L_{w_2+\epsilon}^4 + S_{w_2}^+} - e^{2\pi i(a+b+c+d)} \int_{L_{w_2-\epsilon}^2} - e^{2\pi id} \int_{L_{w_2-\epsilon}^1 + S_{w_2}^+ + L_{w_2+\epsilon}^4} \\ & \left. - \int_{L_{w_2+\epsilon}^3} + \int_{S_{w_2}^+} \right\} v^a (v - w_1)^b (v - w_2)^c (1 - v)^d dv \end{aligned}$$

and

$$\begin{aligned} R_{1-}(w_1, w_2) = & \frac{e^{2\pi ia} - 1}{e^{2\pi i(a+b+c)} - 1} \left\{ \int_{S_{w_2}^+ + L_{w_2-\epsilon}^1} + e^{2\pi i(a+b+c)} \int_{L_{w_2-\epsilon}^2 + S_{w_2}^- + L_{w_2+\epsilon}^3} \right. \\ & + e^{2\pi i(a+b+c+d)} \int_{L_{w_2+\epsilon}^4 - S_{w_2}^- - L_{w_2-\epsilon}^2} - e^{2\pi id} \int_{L_{w_2-\epsilon}^1 + S_{w_2}^+ + L_{w_2+\epsilon}^4} - \int_{L_{w_2+\epsilon}^3} \left. \right\} \\ & \times v^a (v - w_1)^b (v - w_2)^c (1 - v)^d dv, \end{aligned}$$

where the principal branch is used for all powers. A computation gives

$$\begin{aligned} Q_{1-}(w_1, w_2) - R_{1-}(w_1, w_2) = & \frac{e^{2\pi i(a+b)}(e^{2\pi ia} - 1)(e^{2\pi id} - 1)}{(e^{2\pi i(a+b)} - 1)(e^{2\pi i(a+b+c)} - 1)} \\ & \times \left\{ (1 - e^{2\pi ic}) \int_{L_{w_2-\epsilon}^1} + e^{2\pi i(a+b+c)} (1 - e^{2\pi ic}) \int_{L_{w_2-\epsilon}^2} + e^{2\pi ic} (e^{2\pi i(a+b)} - 1) \int_{S_{w_2}^+ + S_{w_2}^-} \right\} \\ & \times v^a (v - w_1)^b (v - w_2)^c (1 - v)^d dv, \quad 0 < w_1 < w_2 < 1. \end{aligned}$$

Using the identity

$$(v - w_2)^c = \begin{cases} e^{i\pi c}(w_2 - v)^c, & v \in S_{w_2}^+ \cup L_{w_2-\epsilon}^1, \\ e^{-i\pi c}(w_2 - v)^c, & v \in S_{w_2}^- \cup L_{w_2-\epsilon}^2, \end{cases}$$

we can write this as

$$\begin{aligned} Q_{1-}(w_1, w_2) - R_{1-}(w_1, w_2) &= \frac{e^{2\pi i(a+b)}(e^{2\pi ia} - 1)(e^{2\pi id} - 1)e^{i\pi c}}{(e^{2\pi i(a+b)} - 1)(e^{2\pi i(a+b+c)} - 1)} \left\{ (1 - e^{2i\pi c}) \int_{L_{w_2-\epsilon}^1} \right. \\ &\quad \left. + e^{2\pi i(a+b)}(1 - e^{2i\pi c}) \int_{L_{w_2-\epsilon}^2} + e^{2\pi ic}(e^{2\pi i(a+b)} - 1) \int_{S_{w_2}^+} + (e^{2\pi i(a+b)} - 1) \int_{S_{w_2}^-} \right\} \\ &\quad \times v^a(v - w_1)^b(w_2 - v)^c(1 - v)^d dv, \quad 0 < w_1 < w_2 < 1. \end{aligned}$$

That is,

$$\begin{aligned} Q_{1-}(w_1, w_2) - R_{1-}(w_1, w_2) &= \frac{e^{2\pi i(a+b)}(e^{2\pi ia} - 1)(e^{2\pi id} - 1)e^{i\pi c}}{(e^{2\pi i(a+b)} - 1)(e^{2\pi i(a+b+c)} - 1)} \\ &\quad \times \int_{w_2-\epsilon}^{(0+, w_2+, 0-, w_2-)} v^a(v - w_1)^b(w_2 - v)^c(1 - v)^d dv, \quad 0 < w_1 < w_2 < 1, \end{aligned}$$

where w_1 lies inside the contour in the same component as 0. Applying the change of variables $s = \frac{v}{w_2}$, which maps the interval $(0, w_2)$ to the interval $(0, 1)$, we obtain

$$\begin{aligned} Q_{1-}(w_1, w_2) - R_{1-}(w_1, w_2) &= \frac{e^{2\pi i(a+b)}(e^{2\pi ia} - 1)(e^{2\pi id} - 1)e^{i\pi c}}{(e^{2\pi i(a+b)} - 1)(e^{2\pi i(a+b+c)} - 1)} w_2^{a+c+1} \\ &\quad \times \int_A^{(0+, 1+, 0-, 1-)} s^a(sw_2 - w_1)^b(1 - s)^c(1 - sw_2)^d ds, \quad 0 < w_1 < w_2 < 1, \end{aligned}$$

where $A \in (0, 1)$ is so large that $Aw_2 - w_1 > 0$. Equation (10.28) follows. \square

The identity (10.26) can be used to determine the asymptotics of $F(w_1, w_2)$ as $w_1 \rightarrow 0$ and $w_2 \rightarrow 0$ to all orders.

10.6 Asymptotics of F as $w_1 \rightarrow 0$ and $w_2 \rightarrow 1$

We finally consider the asymptotics of $F(w_1, w_2)$ in the sector where $w_1 \rightarrow 0$ and $w_2 \rightarrow 1$. Assuming that $a + b, c + d \notin \mathbb{Z}$, we define two functions $\tilde{Q}_1 : \mathcal{D}_1 \rightarrow \mathbb{C}$ and $T_1 : \mathcal{D}_0 \rightarrow \mathbb{C}$ as follows. We define \tilde{Q}_1 by

$$\tilde{Q}_1(w_1, w_2) = \frac{e^{2\pi ia} - 1}{e^{2\pi i(a+b)} - 1} \int_A^{(0+, 1+, 0-, 1-)} v^a(v - w_1)^b(w_2 - v)^c(1 - v)^d dv, \quad (w_1, w_2) \in \mathcal{D}_1,$$

where $A \in (0, 1)$, w_1 lies inside the contour in the same component as 0, and w_2 lies outside the contour. Then

$$Q_1(w_1, w_2) = \rho(w_2)\tilde{Q}_1(w_1, w_2), \quad (w_1, w_2) \in \mathcal{D}_0 \cap \mathcal{D}_1,$$

where ρ is given by (10.8). For $0 < \operatorname{Re} w_1 < \operatorname{Re} w_2 < 1$ with $\operatorname{Im} w_1 < 0$ and $\operatorname{Im} w_2 > 0$, we define $T_1(w_1, w_2)$ by

$$T_1(w_1, w_2) = \frac{(e^{2\pi ia} - 1)(e^{2\pi id} - 1)}{(e^{2\pi i(a+b)} - 1)(e^{2\pi i(c+d)} - 1)} \int_A^{(0+, 1+, 0-, 1-)} v^a (v - w_1)^b (w_2 - v)^c (1 - v)^d dv,$$

where w_1 lies inside the contour in the same component as 0, w_2 lies inside the contour in the same component as 1, and we assume that $\operatorname{Re} w_1 < A < \operatorname{Re} w_2$. We then use analytic continuation to extend T_1 to all of \mathcal{D}_1 .

Lemma 10.4. *Suppose $a + b, c + d \notin \mathbb{Z}$. Then*

$$\tilde{Q}_1(w_1, w_2) = T_1(w_1, w_2) + (w_2 - 1)^{c+d+1} P_2(w_1, w_2), \quad (w_1, w_2) \in \mathcal{D}_1. \quad (10.29)$$

Proof. By analyticity, it is enough to show that

$$\tilde{Q}_{1*}(w_1, w_2) = T_{1*}(w_1, w_2) + |1 - w_2|^{c+d+1} e^{i\pi(c+d+1)} P_{2*}(w_1, w_2), \quad 0 < w_1 < w_2 < 1, \quad (10.30)$$

where $\tilde{Q}_{1*}(w_1, w_2) := \tilde{Q}_1(w_1 - i0, w_2 + i0)$ etc.

Let $0 < w_1 < w_2 < 1$ and let $0 < \epsilon < \frac{1}{2} \min\{w_1, w_2 - w_1, 1 - w_2\}$. Then

$$\begin{aligned} \tilde{Q}_{1*}(w_1, w_2) = & \frac{e^{2\pi ia} - 1}{e^{2\pi i(a+b)} - 1} \left\{ \int_{L_{w_2-\epsilon}^1} + e^{2\pi i(a+b)} \int_{L_{w_2-\epsilon}^2 + S_{w_2}^- + L_{w_2+\epsilon}^3} + e^{2\pi i(a+b+c+d)} \int_{L_{w_2+\epsilon}^4} \right. \\ & - e^{2\pi i(a+b+d)} \int_{S_{w_2}^- + L_{w_2-\epsilon}^2} + e^{2\pi id} \int_{-L_{w_2-\epsilon}^1 + S_{w_2}^-} - e^{2\pi i(c+d)} \int_{L_{w_2+\epsilon}^4} \\ & \left. - \int_{L_{w_2+\epsilon}^3} - \int_{S_{w_2}^-} \right\} v^a (v - w_1)^b (w_2 - v)^c (1 - v)^d dv, \end{aligned}$$

and

$$\begin{aligned} T_{1*}(w_1, w_2) = & \frac{(e^{2\pi ia} - 1)(e^{2\pi id} - 1)}{(e^{2\pi i(a+b)} - 1)(e^{2\pi i(c+d)} - 1)} \left\{ \int_{L_{w_2-\epsilon}^1} + e^{2\pi i(a+b)} \int_{L_{w_2-\epsilon}^2 + S_{w_2}^- + L_{w_2+\epsilon}^3} \right. \\ & + e^{2\pi i(a+b+c+d)} \int_{L_{w_2+\epsilon}^4 + S_{w_2}^+ - L_{w_2-\epsilon}^2} - e^{2\pi i(c+d)} \int_{L_{w_2-\epsilon}^1 + S_{w_2}^+ + L_{w_2+\epsilon}^4} + \int_{-L_{w_2+\epsilon}^3 - S_{w_2}^-} \left. \right\} \\ & \times v^a (v - w_1)^b (w_2 - v)^c (1 - v)^d dv. \end{aligned}$$

It follows that

$$\begin{aligned} \tilde{Q}_{1*}(w_1, w_2) - T_{1*}(w_1, w_2) = & \frac{e^{2\pi ia} - 1}{1 - e^{-2i\pi(c+d)}} \left\{ (1 - e^{-2i\pi c}) \int_{L_{w_2+\epsilon}^3} \right. \\ & + e^{2\pi id} (e^{2i\pi c} - 1) \int_{L_{w_2+\epsilon}^4} + (1 - e^{2\pi id}) \int_{S_{w_2}^+ + S_{w_2}^-} \left. \right\} \\ & \times v^a (v - w_1)^b (w_2 - v)^c (1 - v)^d dv, \quad 0 < w_1 < w_2 < 1. \end{aligned}$$

Using the identity

$$(w_2 - v)^c = \begin{cases} e^{-i\pi c}(v - w_2)^c, & v \in S_{w_2}^+ \cup L_{w_2+\epsilon}^4, \\ e^{i\pi c}(v - w_2)^c, & v \in S_{w_2}^- \cup L_{w_2+\epsilon}^3, \end{cases}$$

we can write this as

$$\begin{aligned} \tilde{Q}_{1*}(w_1, w_2) - T_{1*}(w_1, w_2) &= \frac{(e^{2\pi ia} - 1)e^{-i\pi c}}{1 - e^{-2i\pi(c+d)}} \left\{ (e^{2i\pi c} - 1) \int_{L_{w_2+\epsilon}^3} + e^{2\pi id}(e^{2i\pi c} - 1) \int_{L_{w_2+\epsilon}^4} \right. \\ &\quad \left. + (1 - e^{2\pi id}) \int_{S_{w_2}^+} + e^{2i\pi c}(1 - e^{2\pi id}) \int_{S_{w_2}^-} \right\} v^a(v - w_1)^b(v - w_2)^c(1 - v)^d dv \\ &= \frac{(e^{2\pi ia} - 1)e^{-i\pi c}}{1 - e^{-2i\pi(c+d)}} \int_{w_2+\epsilon}^{(w_2+, 1+, w_2-, 1-)} v^a(v - w_1)^b(v - w_2)^c(1 - v)^d dv, \quad 0 < w_1 < w_2 < 1, \end{aligned}$$

where w_1 lies exterior to the Pochhammer contour. Performing the change of variables $s = \frac{v-w_2}{1-w_2}$, which maps the interval $(w_2, 1)$ to the interval $(0, 1)$, we obtain

$$\begin{aligned} \tilde{Q}_{1*}(w_1, w_2) - T_{1*}(w_1, w_2) &= \frac{(e^{2\pi ia} - 1)e^{-i\pi c}}{1 - e^{-2i\pi(c+d)}} (1 - w_2)^{c+d+1} \\ &\times \int_A^{(0+, 1+, 0-, 1-)} (w_2 + s(1 - w_2))^a (w_2 + s(1 - w_2) - w_1)^b s^c (1 - s)^d ds, \quad 0 < w_1 < w_2 < 1, \end{aligned}$$

where $A \in (0, 1)$. Since

$$\frac{(e^{2\pi ia} - 1)e^{-i\pi c}}{1 - e^{-2i\pi(c+d)}} = \frac{e^{i\pi(a+d)} \sin(a\pi)}{\sin(\pi(d+c))},$$

equation (10.30) follows from (10.16). \square

Recalling (10.22), equation (10.29) yields

$$F(w_1, w_2) = \rho(w_2)[T_1(w_1, w_2) + (w_2 - 1)^{c+d+1}P_2(w_1, w_2)] + w_1^{a+b+1}Q_2(w_1, w_2) \quad (10.31)$$

for $(w_1, w_2) \in \mathcal{D}_0 \cap \mathcal{D}_1$. The identity (10.31) can be used to find the asymptotics of $F(w_1, w_2)$ as $w_1 \rightarrow 0$ and $w_2 \rightarrow 1$.

10.7 Some basic estimates

As explained above, the identities (10.17), (10.22), (10.26), and (10.31) can be used to determine the asymptotics to all orders of $F(w_1, w_2)$ as one or both of the points w_1, w_2 approach 0 or 1. However, for the purposes of establishing Lemma 6.2 (the result relevant for the $\text{SLE}_\kappa(2)$ Green's function), we only need some leading and subleading estimates on F . These estimates can be derived from the identities (10.17), (10.22), (10.26), and (10.31) together with a number of bounds on the functions P_1, Q_j, R_j, T_1, F . We collect the required bounds in the next lemma.

If $w \in \mathbb{C}$ and A is a subset of \mathbb{C} , we write $\text{dist}(w, A)$ for the Euclidean distance from w to A ; we write $\text{dist}(w, A \cup \{\infty\}) > \epsilon$ to indicate that $\text{dist}(w, A) > \epsilon$ and $|w| < 1/\epsilon$.

Lemma 10.5. *Suppose $a, b \geq 0$ and $c, d \leq 0$ satisfy $a, d, a + b, c + d, a + b + c \notin \mathbb{Z}$. Let $\epsilon > 0$. Then the following estimates hold:*

- (a) $|P_1(w_1, w_2)| \leq C$ and $|P_1(w_1, w_2) - P_1(w_1, 1)| \leq C|w_2 - 1|$ uniformly for all $(w_1, w_2) \in \mathbb{C}^2$ such that $\text{dist}(w_1, \{0, 1, \infty\}) > \epsilon$ and $|w_2 - 1| < 1 - \epsilon$.
- (b) $|Q_1(w_1, w_2)| \leq C|w_2|^c$ uniformly for all $(w_1, w_2) \in \mathbb{C}^2$ such that $|w_1| < 1 - \epsilon$ and $\text{dist}(w_2, \{0, 1\}) > \epsilon$.
- (c) $|Q_2(w_1, w_2)| \leq C|w_1|^c$ uniformly for all $(w_1, w_2) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ such that $|w_1| < 1 - \epsilon$ and $\text{dist}(\frac{w_2}{w_1}, \{0, 1\}) > \epsilon$.
- (d) $|R_1(w_1, w_2)| \leq C$ and $|R_1(w_1, w_2) - R_1(0, 0)| \leq C(|w_1| + |w_2|)$ uniformly for all $(w_1, w_2) \in \mathbb{C}^2$ such that $|w_1| < 1 - \epsilon$ and $|w_2| < 1 - \epsilon$.
- (e) $|R_2(w_1, w_2)| \leq C|w_2|^b$ uniformly for all $(w_1, w_2) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ such that $\text{dist}(\frac{w_2}{w_1}, \{0, 1\}) > \epsilon$ and $|w_2| < 1 - \epsilon$.
- (f) $|T_1(w_1, w_2)| \leq C$ uniformly for all $(w_1, w_2) \in \mathbb{C}^2$ such that $|w_1| < 1 - \epsilon$ and $\text{dist}(w_2, \{0, \infty\}) > \epsilon$.
- (g) $|F(w_1, w_2)| \leq C|w_2|^c$ uniformly for all $(w_1, w_2) \in \mathbb{C}^2$ such that $\text{dist}(w_1, \{0, 1, \infty\}) > \epsilon$ and $|w_2| > 1 + \epsilon$.

Proof. The estimates follow easily from the definitions of the functions P_1, Q_j, R_j, T_1, F . \square

Remark 10.6. If (w_1, w_2) lies on a branch cut, the bounds in Lemma 10.5 should be interpreted as saying that both the left and right boundary values obey the bounds.

11 Differential equations

In this section we briefly discuss our results from the point of view of differential equations. We know from the proofs of Theorem 2.1 and Theorem 2.6 that the observables are smooth (so Itô's formula can be applied), and this makes it easy to write down differential equations for them using standard methods. We consider only the case of the Green's function for a system of commuting SLEs started from (ξ^1, ξ^2) with $\xi^1 < \xi^2$. We write the Green's function as

$$\lim_{\epsilon \rightarrow 0} \epsilon^{d-2} \mathbf{P}(\Upsilon_\infty(z) \leq \epsilon) = G(x, y, \xi^1, \xi^2), \quad (z = x + iy). \quad (11.1)$$

Scale invariance of SLE implies that G is scale invariant in the sense that

$$G(\lambda x, \lambda y, \lambda \xi^1, \lambda \xi^2) = \lambda^{d-2} G(x, y, \xi^1, \xi^2), \quad \lambda > 0,$$

and we can write

$$G(x, y, \xi^1, \xi^2) = y^{d-2} H(x, y, \xi^1, \xi^2), \quad (11.2)$$

where the function H is homogeneous:

$$H(\lambda x, \lambda y, \lambda \xi^1, \lambda \xi^2) = H(x, y, \xi^1, \xi^2), \quad \lambda > 0. \quad (11.3)$$

Proposition 11.1. *Let $\kappa \leq 4$ and write $G(x, y, \xi^1, \xi^2)$ for the Green's function for a system of commuting SLE_κ . Then the function H defined by the relation (11.2) satisfies the following two PDEs:*

$$\begin{aligned} \partial_{\xi^j}^2 H + \frac{2a(x - \xi^j)}{y^2 + (x - \xi^j)^2} \partial_x H - \frac{2ay}{y^2 + (x - \xi^j)^2} \partial_y H + \frac{2a}{\xi^1 - \xi^2} \partial_{\xi^1} H + \frac{2a}{\xi^2 - \xi^1} \partial_{\xi^2} H \\ + \frac{\beta y^2}{(y^2 + (x - \xi^j)^2)^2} H = 0, \quad j = 1, 2, \end{aligned} \quad (11.4)$$

where $(x, y, \xi^1, \xi^2) \in \mathbb{R}^4$ with $y > 0$ and $\xi^1 < \xi^2$.

Proof. This follows from the smoothness of G using Itô's formula together with Loewner's equation. We omit the details. \square

Remark 11.2. Schramm's probability satisfies the same PDEs without the last term on the left-hand side.

In addition to the homogeneity property (11.3), the function H is also translation invariant in the x -direction, i.e.,

$$H(x, y, \xi^1, \xi^2) = H(x + \lambda, y, \xi^1 + \lambda, \xi^2 + \lambda), \quad \lambda \in \mathbb{R}. \quad (11.5)$$

Using the two symmetries (11.3) and (11.5), we can reduce the PDEs in (11.4) from four to two dimensions. In fact, one can prove that the function $h(x, \xi) := H(x, 1, -\xi, \xi)$ satisfies an *elliptic* PDE. Note, however, that we are crucially using the smoothness to draw this conclusion. Working from (11.1) we can only derive the PDEs formally without additional arguments establishing smoothness.

A Estimates for Schramm's formula

A.1 Properties of the function $J(z, \xi)$

Recall that we defined $J(z, \xi)$ in (5.2) by

$$J(z, \xi) = \int_{\bar{z}}^z (u - z)^\alpha (u - \bar{z})^{\alpha-2} u^{-\frac{\alpha}{2}} (u - \xi)^{-\frac{\alpha}{2}} du, \quad z \in \mathbb{H}, \quad \xi > 0,$$

where the contour from \bar{z} to z passes to the right of ξ , see Figure 1.

Lemma A.1. *The function $J(z, \xi)$ defined in (5.2) is a well-defined smooth function of $(z, \xi) \in \mathbb{H} \times (0, \infty)$.*

Proof. Since $\alpha > 1$, the integral defining $J(z, \xi)$ is convergent for each $z \in \mathbb{H}$ and each $\xi > 0$. To prove the smoothness of J , we first assume that $\alpha > 1$ is an integer. In this case the integral in (5.2) can be computed explicitly in terms of logarithms and powers of z , \bar{z} , $z - \xi$, and $\bar{z} - \xi$ (see Section 5.2 for the case $\alpha = 2$). Hence $J(z, \xi)$ is smooth for $(z, \xi) \in \mathbb{H} \times (0, \infty)$.

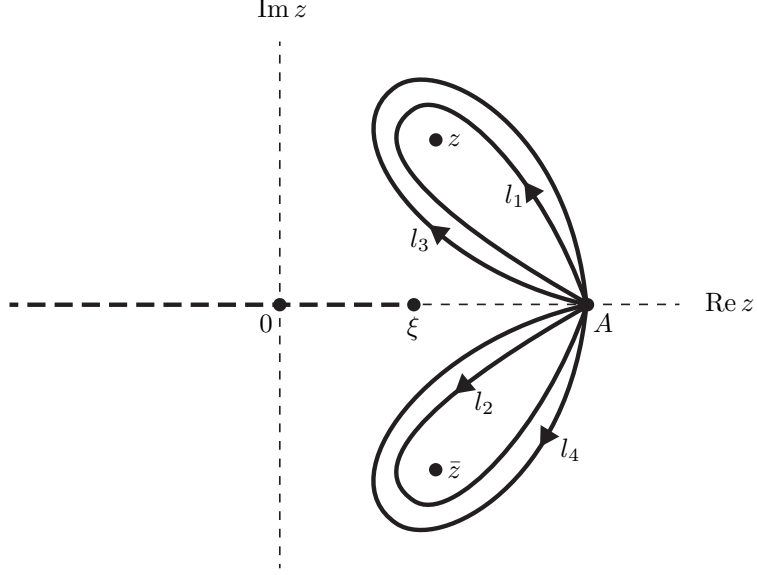


Figure 10. The integration contour in (A.1) is the composition of the four loops l_j , $j = 1, \dots, 4$, based at the point $A > \xi$.

Assume $\alpha > 1$ is not an integer. Then, fixing a basepoint $A > \xi$, we can rewrite the expression (5.2) for $J(z, \xi)$ as

$$J(z, \xi) = \frac{1}{(1 - e^{2i\pi\alpha})^2} \int_A^{(z+, \bar{z}+, z-, \bar{z}-)} (u - z)^\alpha (u - \bar{z})^{\alpha-2} u^{-\frac{\alpha}{2}} (u - \xi)^{-\frac{\alpha}{2}} du, \quad z \in \mathbb{H}, \quad \xi > 0, \quad (\text{A.1})$$

where the integration contour is the composition of four loops $\{l_j\}_1^4$ based at A (see Figure 10) and the integrand is evaluated using analytic continuation along the contour. More precisely, the loop l_1 encircles z once in the counterclockwise direction, l_2 encircles \bar{z} once in the counterclockwise direction, l_3 encircles z once in the clockwise direction, and l_4 encircles \bar{z} once in the clockwise direction. On the first half of l_1 , the principal branch is used, but as the contour l_1 encircles z in the counterclockwise direction, the power $(u - z)^\alpha$ in the integrand picks up an additional factor of $e^{2i\pi\alpha}$ with respect to the principal branch; then, as l_2 encircles \bar{z} in the counterclockwise direction, the power $(u - \bar{z})^{\alpha-2}$ in the integrand picks up the factor $e^{2i\pi(\alpha-2)}$ and so on. Collapsing the contour onto a single path from \bar{z} to z and collecting the exponential factors, we see that (A.1) reduces to (5.2). Since the contour in (A.1) avoids the branch points, the integral in (A.1) can be differentiated an unlimited number of times with respect to z, \bar{z} , and ξ . This completes the proof of the lemma. \square

Lemma A.2. The function $J(z, \xi)$ defined in (5.2) satisfies the following estimates:

$$|J(z, \xi)| \leq C|z - \xi|^{\alpha-1}, \quad z \in \mathbb{H}, \quad \xi > 0, \quad (\text{A.2a})$$

$$|J(z, \xi)| \leq C|x|^{-\frac{\alpha}{2}}|x - \xi|^{-\frac{\alpha}{2}}y^{2\alpha-1}, \quad x > \xi, \quad y > 0, \quad \xi > 0. \quad (\text{A.2b})$$

$$|\operatorname{Re} J(z, \xi)| \leq C y |z|^{\alpha-2}, \quad |z| \geq 2\xi, \quad z \in \mathbb{H}, \quad \xi > 0, \quad (\text{A.2c})$$

where $z = x + iy$.

Proof. To prove (A.2a), we let $z = \xi + re^{i\theta}$ and choose the following parametrization of the integration contour in (5.2) (see Figure 11):

$$u = \xi + re^{i\varphi}, \quad -\theta \leq \varphi \leq \theta. \quad (\text{A.3})$$

This yields after simplification

$$J(\xi + re^{i\theta}, \xi) = ir^{\frac{3\alpha}{2}-1} \int_{-\theta}^{\theta} (e^{i\varphi} - e^{i\theta})^{\alpha} (e^{i\varphi} - e^{-i\theta})^{\alpha-2} (\xi + re^{i\varphi})^{-\frac{\alpha}{2}} e^{i\varphi(1-\frac{\alpha}{2})} d\varphi, \\ r > 0, \quad \theta \in (0, \pi), \quad \xi > 0. \quad (\text{A.4})$$

It follows that

$$|J(\xi + re^{i\theta}, \xi)| \leq r^{\frac{3\alpha}{2}-1} \int_{-\theta}^{\theta} |e^{i\varphi} - e^{i\theta}|^{\alpha} |e^{i\varphi} - e^{-i\theta}|^{\alpha-2} |\xi + re^{i\varphi}|^{-\frac{\alpha}{2}} d\varphi.$$

Since

$$|\xi + re^{i\varphi}| \geq \frac{|re^{i\varphi} + r|}{2},$$

for all $r > 0$, $\varphi \in (-\pi, \pi)$, and $\xi > 0$, we obtain the estimate

$$|J(\xi + re^{i\theta}, \xi)| \leq Cr^{\alpha-1} \int_{-\theta}^{\theta} |e^{i\varphi} - e^{i\theta}|^{\alpha} |e^{i\varphi} - e^{-i\theta}|^{\alpha-2} |e^{i\varphi} + 1|^{-\frac{\alpha}{2}} d\varphi.$$

The integral remains bounded as $\theta \uparrow \pi$, because $\frac{3\alpha}{2} - 2 > -1$. Thus we arrive at

$$|J(\xi + re^{i\theta}, \xi)| \leq Cr^{\alpha-1}, \quad r > 0, \quad \theta \in (0, \pi), \quad \xi > 0,$$

which is (A.2a).

To prove (A.2b), we let $z = x + iy$ in (5.2) and use the parametrization $u = x + is$, $-y \leq s \leq y$, of the contour from \bar{z} to z . Assuming that $x > \xi$, this yields

$$J(z, \xi) = \int_{-y}^y (is - iy)^{\alpha} (is + iy)^{\alpha-2} (x + is)^{-\frac{\alpha}{2}} (x - \xi + is)^{-\frac{\alpha}{2}} i ds. \quad (\text{A.5})$$

It follows that

$$|J(z, \xi)| \leq \int_{-y}^y |s - y|^{\alpha} |s + y|^{\alpha-2} |x + is|^{-\frac{\alpha}{2}} |x - \xi + is|^{-\frac{\alpha}{2}} ds \\ \leq |x|^{-\frac{\alpha}{2}} |x - \xi|^{-\frac{\alpha}{2}} (2y)^{\alpha} \int_{-y}^y |s + y|^{\alpha-2} ds \\ \leq C |x|^{-\frac{\alpha}{2}} |x - \xi|^{-\frac{\alpha}{2}} y^{2\alpha-1}, \quad x > \xi, \quad y > 0, \quad \xi > 0.$$

This proves (A.2b).

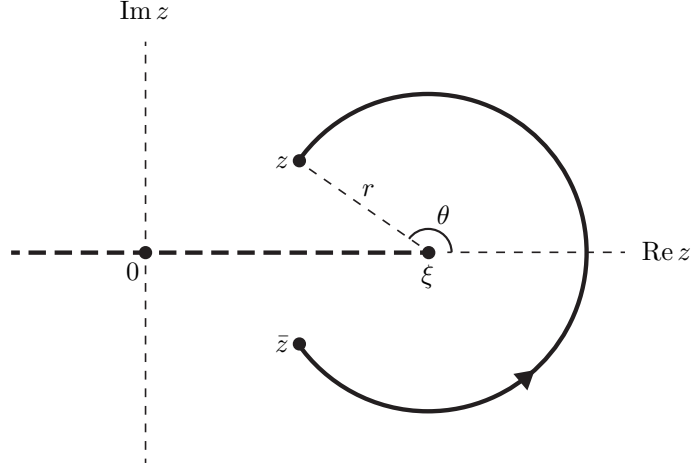


Figure 11. The contour from $\bar{z} = \xi + re^{-i\theta}$ to $z = \xi + re^{i\theta}$ defined in equation (A.3).

To prove (A.2c), we note that if $f(u)$ is an analytic function, then

$$\overline{\int_{\bar{z}}^z f(u) du} = - \int_{\bar{z}}^z \overline{f(\bar{u})} du. \quad (\text{A.6})$$

Hence

$$\begin{aligned} 2\text{Re } J(z, \xi) &= \int_{\bar{z}}^z (u - z)^\alpha (u - \bar{z})^{\alpha-2} u^{-\frac{\alpha}{2}} (u - \xi)^{-\frac{\alpha}{2}} du \\ &\quad - \int_{\bar{z}}^z (u - z)^{\alpha-2} (u - \bar{z})^\alpha u^{-\frac{\alpha}{2}} (u - \xi)^{-\frac{\alpha}{2}} du. \end{aligned}$$

Since

$$(u - z)^2 - (u - \bar{z})^2 = -4iy(u - x),$$

this can be written as

$$\text{Re } J(z, \xi) = -2iy \int_{\bar{z}}^z (u - z)^{\alpha-2} (u - \bar{z})^{\alpha-2} u^{-\frac{\alpha}{2}} (u - \xi)^{-\frac{\alpha}{2}} (u - x) du. \quad (\text{A.7})$$

Assuming that $z = re^{i\theta}$ satisfies $|z| > \xi$ and adopting the parametrization

$$u = re^{i\varphi}, \quad -\theta \leq \varphi \leq \theta,$$

of the integration contour from \bar{z} to z , we arrive at

$$\begin{aligned} \text{Re } J(re^{i\theta}, \xi) &= 2yr^{\frac{3\alpha}{2}-2} \int_{-\theta}^{\theta} (e^{i\varphi} - e^{i\theta})^{\alpha-2} (e^{i\varphi} - e^{-i\theta})^{\alpha-2} (re^{i\varphi} - \xi)^{-\frac{\alpha}{2}} \\ &\quad \times (e^{i\varphi} - \cos \theta) e^{i\varphi(1-\frac{\alpha}{2})} d\varphi. \end{aligned}$$

In view of the estimate

$$|re^{i\varphi} - \xi| \geq \frac{r}{2}$$

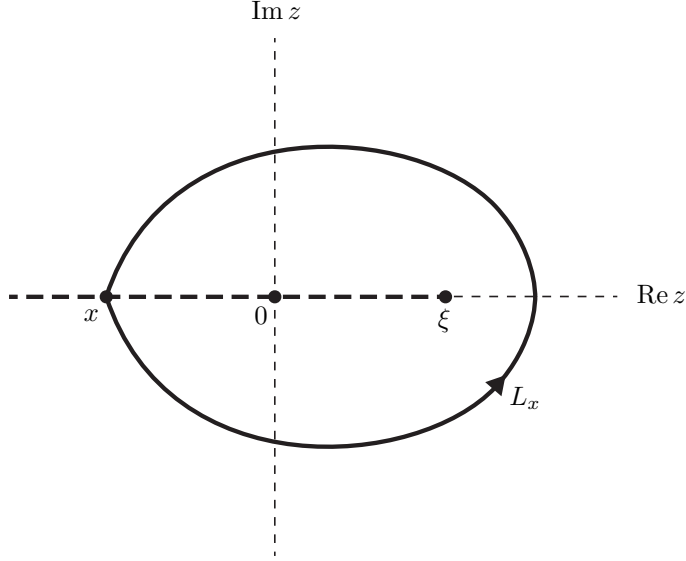


Figure 12. The integration contour L_x in (A.8) is a loop which encloses ξ in the counterclockwise direction.

valid for all $r \geq 2\xi$, $\varphi \in (0, \pi)$, and $\xi > 0$, this implies

$$|\operatorname{Re} J(re^{i\theta}, \xi)| \leq C y r^{\alpha-2} \int_{-\theta}^{\theta} |e^{i\varphi} - e^{i\theta}|^{\alpha-2} |e^{i\varphi} - e^{-i\theta}|^{\alpha-2} |e^{i\varphi} - \cos \theta| d\varphi$$

$$r \geq 2\xi, \quad \theta \in (0, \pi), \quad \xi > 0.$$

Since $2\alpha - 3 > -1$, the integral remains bounded as $\theta \uparrow \pi$. This proves (A.2c). \square

We next extend the function $J(z, \xi)$ continuously to all of $\bar{\mathbb{H}} \times (0, \infty)$. As suggested by (5.2), we define $J(x, \xi)$ for $x \in \mathbb{R}$ by

$$J(x, \xi) = \begin{cases} 0, & x \geq \xi, \\ \int_{L_x} (u-x)^{2\alpha-2} u^{-\frac{\alpha}{2}} (u-\xi)^{-\frac{\alpha}{2}} du, & x < \xi, \end{cases} \quad (\text{A.8})$$

where the integration contour L_x is a loop starting and ending at x which avoids the branch cut along $(-\infty, \xi)$ and which encloses ξ in the counterclockwise direction, see Figure 12.

Lemma A.3. *For each $\xi > 0$, the function $J(z, \xi)$ defined by (5.2) and (A.8) is a continuous function of $z \in \bar{\mathbb{H}}$.*

Proof. Fix $\xi > 0$. Since $\alpha > 1$, the integral in (A.8) converges for every $x < \xi$ (including $x = 0$). Equation (A.2b) implies that $z \mapsto J(z, \xi)$ is continuous at each point in (ξ, ∞) . Moreover, equation (A.2a) implies that J is continuous at $z = \xi$.

We next show that J is continuous at each point in $(-\infty, 0) \cup (0, \xi)$. Letting $s = \frac{\pi}{\theta}\varphi$ and simplifying, we can write (A.4) as

$$J(\xi + re^{i\theta}, \xi) = \int_{-\pi}^{\pi} g_{r,\theta}(s) ds,$$

where

$$g_{r,\theta}(s) = ir^{\frac{3\alpha}{2}-1} \frac{\theta}{\pi} (e^{\frac{i\theta s}{\pi}} - e^{i\theta})^\alpha (e^{\frac{i\theta s}{\pi}} - e^{-i\theta})^{\alpha-2} (\xi + re^{\frac{i\theta s}{\pi}})^{-\frac{\alpha}{2}} e^{\frac{i\theta s}{\pi}(1-\frac{\alpha}{2})}.$$

Let $\epsilon > 0$. Then

$$\begin{aligned} |g_{r,\theta}(s)| &\leq C |e^{\frac{i\theta s}{\pi}} - e^{i\theta}|^\alpha |e^{\frac{i\theta s}{\pi}} - e^{-i\theta}|^{\alpha-2} \\ &\leq C 2^\alpha \max \left(\left| \theta + \frac{\theta s}{\pi} \right|^{\alpha-2}, \left| \theta + \frac{\theta s}{\pi} - 2\pi \right|^{\alpha-2} \right) \\ &\leq C \max (|\pi + s|^{\alpha-2}, |\pi - s|^{\alpha-2}), \quad s \in (-\pi, \pi), \end{aligned}$$

for all $r \in (\epsilon, \epsilon^{-1})$ with $|r - \xi| > \epsilon$ and all $\theta \in [\frac{\pi}{2}, \pi]$. Since $|\pi \pm s|^{\alpha-2} \in L^1([-\pi, \pi])$, this shows that there exists a function $G(s)$ in $L^1([-\pi, \pi])$ such that

$$|g_{r,\theta}(s)| \leq G(s), \quad s \in (-\pi, \pi), \quad (\text{A.9})$$

for all $r \in (\epsilon, \epsilon^{-1})$ with $|r - \xi| > \epsilon$ and all $\theta \in [\frac{\pi}{2}, \pi]$. Since $\epsilon > 0$ was arbitrary, if the point $x_0 = \xi + r_0 e^{i\pi}$ belongs to $(-\infty, 0) \cup (0, \xi)$, dominated convergence gives

$$\lim_{z \rightarrow x_0} J(z, \xi) = \lim_{\substack{\theta \uparrow \pi \\ r \rightarrow r_0}} \int_{-\pi}^{\pi} g_{r,\theta}(s) ds = \int_{-\pi}^{\pi} g_{r_0,\pi}(s) ds = J(x_0, \xi).$$

This shows that $J(\cdot, \xi)$ is continuous at each point in $(-\infty, 0) \cup (0, \xi)$.

It only remains to show that $J(\cdot, \xi) : \bar{\mathbb{H}} \rightarrow \mathbb{C}$ is continuous at $z = 0$. To prove this, let $c = \frac{\xi}{2}$ and let $z = re^{i\theta}$ with $\theta \in [0, \pi]$ and $r \in (0, c)$. Let γ denote the contour from \bar{z} to z used in the definition (5.2) of $J(z, \xi)$. We write γ as the union of five subcontours as follows (see Figure 13):

$$\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5,$$

where

$$\begin{aligned} \gamma_1 &: \{re^{i\varphi} \mid -\theta \leq \varphi \leq 0\}, & \gamma_2 &: \{u - i0 \mid r \leq u \leq \xi - c\}, \\ \gamma_3 &: \{\xi + ce^{i\varphi} \mid -\pi \leq \varphi \leq \pi\}, & \gamma_4 &: \{u + i0 \mid r \leq u \leq \xi - c\}, \\ \gamma_5 &: \{re^{i\varphi} \mid 0 \leq \varphi \leq \theta\}. \end{aligned} \quad (\text{A.10})$$

Then

$$J(z, \xi) = \sum_{j=1}^5 J_j(z),$$

where

$$J_j(z) = \int_{\gamma_j} (u - z)^\alpha (u - \bar{z})^{\alpha-2} u^{-\frac{\alpha}{2}} (u - \xi)^{-\frac{\alpha}{2}} du, \quad j = 1, \dots, 5.$$

We claim that

$$|J_j(z)| \leq Cr^{\frac{3\alpha}{2}-1}, \quad |z| < c, \quad z \in \mathbb{H}, \quad j = 1, 5. \quad (\text{A.11})$$

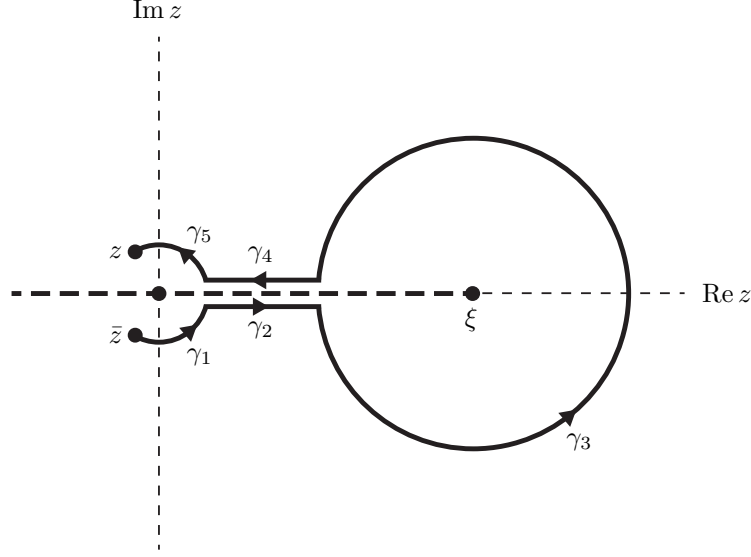


Figure 13. The integration contour γ is the composition of the five subcontours γ_j , $j = 1, \dots, 5$, defined in (A.10).

Indeed, let us consider the case of $J_1(z)$. Since $|u - \xi| \geq c$ for $u \in \gamma_1$, we have

$$|J_1(z)| \leq \int_{\gamma_1} |u - z|^\alpha |u - \bar{z}|^{\alpha-2} r^{-\frac{\alpha}{2}} c^{-\frac{\alpha}{2}} |du|.$$

Moreover, the inequalities

$$|u - \bar{z}| \leq |u - z| \leq 2r$$

are valid for $u \in \gamma_1$. Hence, if $\alpha \geq 2$, then

$$|J_1(z)| \leq C \int_{\gamma_1} |u - z|^{2\alpha-2} r^{-\frac{\alpha}{2}} |du| \leq C \int_{\gamma_1} r^{\frac{3\alpha}{2}-2} |du| \leq Cr^{\frac{3\alpha}{2}-1}.$$

On the other hand, if $1 < \alpha < 2$, then

$$\begin{aligned} |J_1(z)| &\leq C \int_{\gamma_1} (2r)^\alpha |u - \bar{z}|^{\alpha-2} r^{-\frac{\alpha}{2}} |du| \leq Cr^{\frac{\alpha}{2}} \int_0^\theta |re^{-i\varphi} - re^{-i\theta}|^{\alpha-2} r d\varphi \\ &\leq Cr^{\frac{3\alpha}{2}-1} \int_0^\theta \left| \frac{2}{\pi}(\theta - \varphi) \right|^{\alpha-2} d\varphi \leq Cr^{\frac{3\alpha}{2}-1} \frac{\theta^{\alpha-1}}{\alpha-1} \leq Cr^{\frac{3\alpha}{2}-1}. \end{aligned}$$

This proves (A.11) for $J_1(z)$; the proof for $J_5(z)$ is similar.

We next show that

$$\lim_{z \rightarrow 0} J_j(z) = J_j(0), \quad j = 2, 4. \quad (\text{A.12})$$

To establish (A.12), we let $u = r + s$ and write

$$J_2(z) = \int_0^c f_{r,\theta}(s) ds,$$

where

$$f_{r,\theta}(s) = \chi_{[0,c-r]}(s)(s+r-re^{i\theta})^\alpha(s+r-re^{-i\theta})^{\alpha-2}(s+r)^{-\frac{\alpha}{2}}|s+r-\xi|^{-\frac{\alpha}{2}}e^{\frac{\alpha i\pi}{2}}$$

and $\chi_{[0,c-r]}$ denotes the characteristic function of the interval $[0, c-r]$. We will show that there exists a function $F(s)$ in $L^1((0, c))$ such that

$$|f_{r,\theta}(s)| \leq F(s), \quad s \in (0, c), \quad (\text{A.13})$$

for all $r \in (0, c)$ and all $\theta \in [0, \pi]$. Dominated convergence then gives

$$\lim_{z \rightarrow 0} J_2(z) = \lim_{r \rightarrow 0} \int_0^c f_{r,\theta}(s) ds = \int_0^c f_{0,\theta}(s) ds = J_2(0),$$

showing that $J_2(z)$ satisfies (A.12).

In order to prove (A.13), we note that

$$|s+r-re^{i\theta}| = |s+r-re^{-i\theta}| \quad \text{and} \quad |s+r-\xi| \geq c$$

for $s \in (0, c-r)$. This gives

$$|f_{r,\theta}(s)| \leq |s+r-re^{i\theta}|^{2\alpha-2}(s+r)^{-\frac{\alpha}{2}}c^{-\frac{\alpha}{2}}.$$

Using the inequalities

$$|s+r-re^{i\theta}| \leq s+2r, \quad s+r \geq \frac{s+2r}{2},$$

we find

$$|f_{r,\theta}(s)| \leq |s+2r|^{\frac{3\alpha}{2}-2}(2/c)^{\frac{\alpha}{2}}, \quad s \in (0, c), \quad r \in (0, c), \quad \theta \in [0, \pi].$$

Since

$$|s+2r|^{\frac{3\alpha}{2}-2} \leq \begin{cases} (3c)^{\frac{3\alpha}{2}-2}, & \alpha \geq \frac{4}{3}, \\ s^{\frac{3\alpha}{2}-2}, & 1 < \alpha < \frac{4}{3}, \end{cases}$$

we deduce that (A.13) holds with

$$F(s) = (2/c)^{\frac{\alpha}{2}} \max((3c)^{\frac{3\alpha}{2}-2}, s^{\frac{3\alpha}{2}-2}), \quad 0 < s < c.$$

This proves (A.12) for $j = 2$; the proof when $j = 4$ is similar.

Finally, since the integration contour is independent of z , it is easy to see that

$$\lim_{z \rightarrow 0} J_3(z) = J_3(0). \quad (\text{A.14})$$

Since $J(z, \xi) = \sum_{j=1}^5 J_j(z)$, the continuity of $J(z, \xi)$ at $z = 0$ follows from equations (A.11), (A.12), and (A.14). This completes the proof of the lemma. \square

Lemma A.4. *For each $\xi > 0$, the partial derivatives $\partial_x J(z, \xi)$ and $\partial_y J(z, \xi)$ have continuous extensions to $\bar{\mathbb{H}} \setminus \{0, \xi\}$.*

Proof. Fix $\xi > 0$. Defining $W(u)$ by

$$W(u) = u^{-\frac{\alpha}{2}}(u - \xi)^{-\frac{\alpha}{2}}, \quad (\text{A.15})$$

we can write

$$J(z, \xi) = \int_{\bar{z}}^z (u - z)^\alpha (u - \bar{z})^{\alpha-2} W(u) du.$$

An integration by parts gives

$$\begin{aligned} J(z, \xi) &= -\frac{\alpha}{\alpha-1} \int_{\bar{z}}^z (u - z)^{\alpha-1} (u - \bar{z})^{\alpha-1} W(u) du \\ &\quad - \frac{1}{\alpha-1} \int_{\bar{z}}^z (u - z)^\alpha (u - \bar{z})^{\alpha-1} W'(u) du. \end{aligned} \quad (\text{A.16})$$

Differentiating with respect to z and \bar{z} , we find

$$\begin{aligned} J_z(z, \xi) &= \alpha \int_{\bar{z}}^z (u - z)^{\alpha-2} (u - \bar{z})^{\alpha-1} W(u) du \\ &\quad + \frac{\alpha}{\alpha-1} \int_{\bar{z}}^z (u - z)^{\alpha-1} (u - \bar{z})^{\alpha-1} W'(u) du \end{aligned}$$

and

$$\begin{aligned} J_{\bar{z}}(z, \xi) &= \alpha \int_{\bar{z}}^z (u - z)^{\alpha-1} (u - \bar{z})^{\alpha-2} W(u) du \\ &\quad + \int_{\bar{z}}^z (u - z)^\alpha (u - \bar{z})^{\alpha-2} W'(u) du. \end{aligned}$$

Since $\alpha > 1$, these expressions for J_z and $J_{\bar{z}}$ are well-defined for each $z \in \mathbb{R} \setminus \{0, \xi\}$. Repeating the above arguments that led to the continuity of $J(z, \xi)$ at each point $z \in \mathbb{R} \setminus \{0, \xi\}$, we infer that this provides continuous extensions of J_z and $J_{\bar{z}}$ to $\mathbb{H} \setminus \{0, \xi\}$. \square

Lemma A.5. *For each fixed $\xi > 0$, the function $J(z, \xi)$ satisfies*

$$\operatorname{Re} J(x, \xi) = 0, \quad x \in \mathbb{R}, \quad \xi > 0. \quad (\text{A.17})$$

Moreover,

$$\begin{cases} J(x + iy, \xi) = O(1), \\ \operatorname{Re} J(x + iy, \xi) = O(y), \end{cases} \quad y \downarrow 0, \quad x \in \mathbb{R} \setminus \{0, \xi\}, \quad (\text{A.18a})$$

and

$$J(x + iy, \xi) = O(y^{2\alpha-1}), \quad y \downarrow 0, \quad x > \xi, \quad (\text{A.18b})$$

where the error terms are uniform with respect to x in compact subsets of $\mathbb{R} \setminus \{0, \xi\}$.

Proof. Equation (A.17) follows by letting $y \rightarrow 0$ in (A.7). The asymptotic formulas (A.18a) are then a direct consequence of Lemma A.3 and Lemma A.4. Equation (A.18b) follows from (A.2b). \square

A.2 Existence and regularity of P

This section studies the integral in (2.5) predicted to equal Schramm's formula.

Lemma A.6. *The function $\mathcal{M}(z, \xi)$ defined in (2.4) satisfies the following estimates:*

$$|\mathcal{M}(z, \xi)| \leq Cy^{\alpha-2}|z|^{1-\alpha}, \quad z \in \mathbb{H}, \quad \xi > 0, \quad (\text{A.19a})$$

$$|\mathcal{M}(z, \xi)| \leq Cy^{3\alpha-3}|z|^{1-\alpha}|z - \xi|^{1-\alpha}|x|^{-\frac{\alpha}{2}}|x - \xi|^{-\frac{\alpha}{2}}, \quad x > \xi, \quad y > 0, \quad \xi > 0, \quad (\text{A.19b})$$

$$\begin{aligned} |\operatorname{Re} \mathcal{M}(z, \xi)| &\leq Cy^{\alpha-1}|z|^{-\alpha}|z - \xi|^{-\alpha} \left[(|x||x - \xi| + y^2)|z|^{\alpha-2} \right. \\ &\quad \left. + (|x| + |\xi - x|)|z - \xi|^{\alpha-1} \right], \quad |z| \geq 2\xi, \quad z \in \mathbb{H}, \quad \xi > 0, \end{aligned} \quad (\text{A.19c})$$

where $z = x + iy$. In particular, for each fixed $\xi > 0$,

$$\mathcal{M}(z, \xi) = O(|x|^{1-\alpha}), \quad x \rightarrow \pm\infty, \quad y > 0, \quad (\text{A.20a})$$

$$\operatorname{Re} \mathcal{M}(z, \xi) = O(|x|^{-\alpha}), \quad x \rightarrow \pm\infty, \quad y > 0, \quad (\text{A.20b})$$

where the error terms are uniform with respect to y in compact subsets of $(0, \infty)$.

Proof. The estimates (A.19a) and (A.19b) follow immediately from the definition of \mathcal{M} together with the estimates (A.2a) and (A.2b) of Lemma A.2. The estimate (A.19c) follows by applying (A.2a) and (A.2c) to the identity

$$\begin{aligned} \operatorname{Re} \mathcal{M}(z, \xi) &= y^{\alpha-2}|z|^{-\alpha}|z - \xi|^{-\alpha} \operatorname{Re} [\bar{z}(\bar{z} - \xi)J(z, \xi)] \\ &= y^{\alpha-2}|z|^{-\alpha}|z - \xi|^{-\alpha} [(x^2 - x\xi - y^2)\operatorname{Re} J(z, \xi) - y(\xi - 2x)\operatorname{Im} J(z, \xi)]. \end{aligned} \quad (\text{A.21})$$

Finally, the asymptotic equations in (A.20) are an immediate consequence of (A.19a) and (A.19c). \square

Lemma A.7. *The function $\mathcal{M} = \mathcal{M}_1 + i\mathcal{M}_2$ satisfies*

$$\partial_y \mathcal{M}_1(z, \xi) = -\partial_x \mathcal{M}_2(z, \xi), \quad z \in \mathbb{H}, \quad \xi > 0.$$

Proof. Since the statement only involves derivatives with respect to x and y , we can assume that $\xi > 0$ is fixed. We need to prove that $\operatorname{Im} \bar{\partial} \mathcal{M} = 0$. In terms of the function $W(u)$ defined in (A.15) we can write

$$\mathcal{M}(z, \xi) = y^{\alpha-2}z^{-\frac{\alpha}{2}}(z - \xi)^{-\frac{\alpha}{2}}\bar{z}^{1-\frac{\alpha}{2}}(\bar{z} - \xi)^{1-\frac{\alpha}{2}} \int_{\bar{z}}^z (u - z)^\alpha (u - \bar{z})^{\alpha-2} W(u) du.$$

An integration by parts gives

$$\begin{aligned} \mathcal{M}(z, \xi) &= -y^{\alpha-2}z^{-\frac{\alpha}{2}}(z - \xi)^{-\frac{\alpha}{2}}\bar{z}^{1-\frac{\alpha}{2}}(\bar{z} - \xi)^{1-\frac{\alpha}{2}} \frac{1}{\alpha - 1} \\ &\quad \times \left(\alpha \int_{\bar{z}}^z (u - z)^{\alpha-1} (u - \bar{z})^{\alpha-1} W(u) du + \int_{\bar{z}}^z (u - z)^\alpha (u - \bar{z})^{\alpha-1} W'(u) du \right). \end{aligned} \quad (\text{A.22})$$

Differentiating with respect to \bar{z} , we find

$$\begin{aligned}\bar{\partial}\mathcal{M}(z, \xi) &= \left(\frac{i}{2} \frac{\alpha - 2}{y} + \frac{1 - \frac{\alpha}{2}}{\bar{z}} + \frac{1 - \frac{\alpha}{2}}{\bar{z} - \xi} \right) \mathcal{M}(z, \xi) \\ &+ y^{\alpha-2} z^{-\frac{\alpha}{2}} (z - \xi)^{-\frac{\alpha}{2}} \bar{z}^{1-\frac{\alpha}{2}} (\bar{z} - \xi)^{1-\frac{\alpha}{2}} \\ &\times \left(\alpha \int_{\bar{z}}^z (u - z)^{\alpha-1} (u - \bar{z})^{\alpha-2} W(u) du + \int_{\bar{z}}^z (u - z)^{\alpha} (u - \bar{z})^{\alpha-2} W'(u) du \right).\end{aligned}\quad (\text{A.23})$$

The integrals in (A.23) are convergent at the endpoints z and \bar{z} because $\alpha > 1$. Substituting the expression (A.22) for \mathcal{M} into (A.23) and simplifying, we obtain

$$\begin{aligned}\bar{\partial}\mathcal{M}(z, \xi) &= R(z, \xi) \left\{ -i(\alpha - 2)[x(x - \xi) + y^2] \right. \\ &\times \left(\alpha \int_{\bar{z}}^z (u - z)^{\alpha-1} (u - \bar{z})^{\alpha-1} W(u) du + \int_{\bar{z}}^z (u - z)^{\alpha} (u - \bar{z})^{\alpha-1} W'(u) du \right) \\ &+ 2(\alpha - 1)y\bar{z}(\bar{z} - \xi) \\ &\times \left. \left(\alpha \int_{\bar{z}}^z (u - z)^{\alpha-1} (u - \bar{z})^{\alpha-2} W(u) du + \int_{\bar{z}}^z (u - z)^{\alpha} (u - \bar{z})^{\alpha-2} W'(u) du \right) \right\},\end{aligned}\quad (\text{A.24})$$

where the real-valued function $R(z, \xi)$ is given by

$$R(z, \xi) = \frac{1}{2(\alpha - 1)} y^{\alpha-3} |z|^{-\alpha} |z - \xi|^{-\alpha}.$$

To establish the identity $\text{Im } \bar{\partial}\mathcal{M} = 0$ it is enough to show that

$$\frac{\bar{\partial}\mathcal{M} - \overline{\partial\mathcal{M}}}{R} = 0. \quad (\text{A.25})$$

Using (A.6) and (A.24), we can write the left-hand side of (A.25) as

$$\begin{aligned}&-i(\alpha - 2)[x(x - \xi) + y^2] \\ &\times \left(\alpha \int_{\bar{z}}^z (u - z)^{\alpha-1} (u - \bar{z})^{\alpha-1} W(u) du + \int_{\bar{z}}^z (u - z)^{\alpha} (u - \bar{z})^{\alpha-1} W'(u) du \right) \\ &+ 2(\alpha - 1)y\bar{z}(\bar{z} - \xi) \\ &\times \left(\alpha \int_{\bar{z}}^z (u - z)^{\alpha-1} (u - \bar{z})^{\alpha-2} W(u) du + \int_{\bar{z}}^z (u - z)^{\alpha} (u - \bar{z})^{\alpha-2} W'(u) du \right) \\ &-i(\alpha - 2)[x(x - \xi) + y^2] \\ &\times \left(-\alpha \int_{\bar{z}}^z (u - z)^{\alpha-1} (u - \bar{z})^{\alpha-1} W(u) du - \int_{\bar{z}}^z (u - z)^{\alpha-1} (u - \bar{z})^{\alpha} W'(u) du \right) \\ &-2(\alpha - 1)yz(z - \xi) \\ &\times \left(-\alpha \int_{\bar{z}}^z (u - z)^{\alpha-2} (u - \bar{z})^{\alpha-1} W(u) du - \int_{\bar{z}}^z (u - z)^{\alpha-2} (u - \bar{z})^{\alpha} W'(u) du \right).\end{aligned}\quad (\text{A.26})$$

The right-hand side of (A.26) involves eight integrals. Integrating by parts in the third, fourth, and eighth of these integrals and using that the first and fifth integrals cancel, we see that the expression in (A.26) can be written as

$$\begin{aligned}
& i(\alpha - 2)[x(x - \xi) + y^2] \\
& \times \left(- \int_{\bar{z}}^z (u - z)^\alpha (u - \bar{z})^{\alpha-1} W'(u) du + \int_{\bar{z}}^z (u - z)^{\alpha-1} (u - \bar{z})^\alpha W'(u) du \right) \\
& + 2(\alpha - 1)y\bar{z}(\bar{z} - \xi) \\
& \times \left(- \alpha \int_{\bar{z}}^z (u - z)^{\alpha-2} (u - \bar{z})^{\alpha-1} W(u) du - \alpha \int_{\bar{z}}^z (u - z)^{\alpha-1} \frac{(u - \bar{z})^{\alpha-1}}{\alpha - 1} W'(u) du \right. \\
& \quad \left. - \alpha \int_{\bar{z}}^z (u - z)^{\alpha-1} \frac{(u - \bar{z})^{\alpha-1}}{\alpha - 1} W'(u) du - \int_{\bar{z}}^z (u - z)^\alpha \frac{(u - \bar{z})^{\alpha-1}}{\alpha - 1} W''(u) du \right) \\
& - 2(\alpha - 1)yz(z - \xi) \\
& \times \left(- \alpha \int_{\bar{z}}^z (u - z)^{\alpha-2} (u - \bar{z})^{\alpha-1} W(u) du \right. \\
& \quad \left. + \alpha \int_{\bar{z}}^z \frac{(u - z)^{\alpha-1}}{\alpha - 1} (u - \bar{z})^{\alpha-1} W'(u) du + \int_{\bar{z}}^z \frac{(u - z)^{\alpha-1}}{\alpha - 1} (u - \bar{z})^\alpha W''(u) du \right).
\end{aligned} \tag{A.27}$$

A long but straightforward computation shows that the expression in (A.27) equals

$$2\alpha y \int_{\bar{z}}^z \frac{d}{du} \left[(u - z)^\alpha (u - \bar{z})^\alpha u^{-\frac{\alpha}{2}} (u - \xi)^{-\frac{\alpha}{2}} \left(\frac{2x - \xi}{u - z} - \frac{x - \xi}{u} - \frac{x}{u - \xi} \right) \right] du.$$

Since $\alpha > 1$, the fundamental theorem of calculus implies that the integral vanishes. This proves (A.25) and completes the proof of the lemma. \square

Lemma A.8. *The function $P(z, \xi)$ defined in (2.5) is a well-defined smooth function of $(z, \xi) \in \mathbb{H} \times (0, \infty)$.*

Proof. By Lemma A.1, $\mathcal{M} = \mathcal{M}_1 + i\mathcal{M}_2$ is a smooth function of $(z, \xi) \in \mathbb{H} \times (0, \infty)$. Moreover, by equation (A.20), there exists an $\epsilon > 0$ such that $\mathcal{M}_1(x + iy) = O(|x|^{-1-\epsilon})$ and $\mathcal{M}_2(x + iy) = O(|x|^{-\epsilon})$ as $|x| \rightarrow \infty$ uniformly with respect to y in compact subsets of $(0, \infty)$. It follows that the integral in the definition (2.5) of P converges. Furthermore, Lemma A.7 shows that the integral $\int(\mathcal{M}_1 dx - \mathcal{M}_2 dy)$ is independent of the path. We infer that $P(z, \xi)$ can be written as

$$\begin{aligned}
P(z, \xi) &= -\frac{1}{c_\alpha} \int_{\infty}^z \left(\mathcal{M}_1(z', \xi) dx' - \mathcal{M}_2(z', \xi) dy' \right) \\
&= -\frac{1}{c_\alpha} \int_{\infty}^z \operatorname{Re} [\mathcal{M}(z', \xi) dz'], \quad z \in \mathbb{H},
\end{aligned} \tag{A.28}$$

where $z' = x' + iy'$ and the contour of integration runs from $\infty + ic$, $c > 0$, to z . Since $\mathcal{M}(z, \xi)$ is a smooth function of $(z, \xi) \in \mathbb{H} \times (0, \infty)$, so is $P(z, \xi)$. \square

Lemma A.9. *For each $\xi > 0$, the function $\mathcal{M}(z, \xi)$ satisfies*

$$\begin{cases} \mathcal{M}(x + iy, \xi) = O(y^{\alpha-2}), \\ \operatorname{Re} \mathcal{M}(x + iy, \xi) = O(y^{\alpha-1}), \end{cases} \quad y \downarrow 0, \quad x \in \mathbb{R} \setminus \{0, \xi\}, \quad (\text{A.29})$$

and

$$\mathcal{M}(x + iy, \xi) = O(y^{3\alpha-3}), \quad y \downarrow 0, \quad x > \xi,$$

where the error terms are uniform with respect to x in compact subsets of $\mathbb{R} \setminus \{0, \xi\}$.

Proof. This follows from Lemma A.5 and the definition (2.4) of \mathcal{M} . \square

Lemma A.10. *For each $\xi > 0$, the function $z \mapsto P(z, \xi)$ defined in (2.5) has a continuous extension to $\bar{\mathbb{H}} \setminus \{0\}$ which satisfies*

$$P(x, \xi) = \begin{cases} 1, & -\infty < x < 0, \\ 0, & 0 < x < \infty, \end{cases} \quad \xi > 0. \quad (\text{A.30})$$

Proof. Fix $\xi > 0$. The expression (A.28) for P together with Lemma A.9 imply that there exist real constants $\{P_j\}_1^3$ such that the function

$$z \mapsto \begin{cases} P(z, \xi), & z \in \mathbb{H}, \\ P_1, & z \in (-\infty, 0), \\ P_2, & z \in (0, \xi), \\ P_3, & z \in (\xi, \infty), \end{cases}$$

is continuous $\bar{\mathbb{H}} \setminus \{0, \xi\} \rightarrow \mathbb{R}$. Letting z approach $\infty + i0$ in (A.28), we deduce that $P_3 = 0$. It follows that

$$P(\xi + re^{i\theta}, \xi) = -\frac{1}{c_\alpha} \operatorname{Re} \left(ir \int_0^\theta \mathcal{M}(\xi + re^{i\varphi}, \xi) e^{i\varphi} d\varphi \right)$$

for $r > 0$ and $\theta \in (0, \pi)$. In view of the estimate (A.19a), this yields

$$\begin{aligned} |P(\xi + re^{i\theta}, \xi)| &\leq Cr \int_0^\theta |\mathcal{M}(\xi + re^{i\varphi}, \xi)| d\varphi \\ &\leq Cr^{\alpha-1} \int_0^\theta (\sin^{\alpha-2} \varphi) |\xi + re^{i\varphi}|^{1-\alpha} d\varphi, \quad r > 0, \quad \xi > 0. \end{aligned} \quad (\text{A.31})$$

Letting $r \downarrow 0$, we infer that if we set $P(\xi, \xi) = 0$, then $P(z, \xi)$ is continuous at $z = \xi$. Moreover, taking the limit $r \downarrow 0$ in (A.31) with $\theta = \pi$, it follows that $P_2 = 0$.

It only remains to prove that $P_3 = 1$. This will follow if we can show that the normalization constant c_α defined in (2.6) satisfies

$$c_\alpha = -\operatorname{Re} \int_{S_r} \mathcal{M}(z, \xi) dz, \quad r > 0, \quad (\text{A.32})$$

where S_r is a counterclockwise semicircle of radius r centered at 0:

$$S_r : re^{i\varphi}, \quad 0 \leq \varphi \leq \pi.$$

Lemmas A.7 and A.9 show that the right-hand side of (A.32) is independent of $r > 0$. We can therefore evaluate it in the limit as $r \downarrow 0$. Recalling the definition of \mathcal{M} , we write

$$\int_{S_r} \mathcal{M}(z, \xi) dz = \int_0^\pi f_r(\varphi) d\varphi,$$

where

$$f_r(\varphi) = i(\sin^{\alpha-2} \varphi)(re^{i\varphi} - \xi)^{-\frac{\alpha}{2}}(re^{-i\varphi} - \xi)^{1-\frac{\alpha}{2}} J(re^{i\varphi}, \xi).$$

The function J is bounded on each compact subset of $\bar{\mathbb{H}}$ by Lemma A.3. Hence $f_r(\varphi)$ obeys the estimate

$$|f_r(\varphi)| \leq C \sin^{\alpha-2} \varphi, \quad r < \xi/2, \quad \varphi \in [0, \pi].$$

Since the function $\sin^{\alpha-2} \varphi$ belongs to $L^1((0, \pi))$, dominated convergence yields

$$\begin{aligned} -\operatorname{Re} \int_{S_r} \mathcal{M}(z, \xi) dz &= -\operatorname{Re} \lim_{r \rightarrow 0} \int_0^\pi f_r(\varphi) d\varphi = -\operatorname{Re} \int_0^\pi \lim_{r \rightarrow 0} f_r(\varphi) d\varphi \\ &= -\operatorname{Re} \int_0^\pi i(\sin^{\alpha-2} \varphi)(-\xi + i0)^{-\frac{\alpha}{2}}(-\xi - i0)^{1-\frac{\alpha}{2}} J(0, \xi) d\varphi \\ &= -(\operatorname{Im} J(0, \xi)) \xi^{1-\alpha} \int_0^\pi \sin^{\alpha-2} \varphi d\varphi. \end{aligned}$$

But setting $r = \xi$ and $\theta = \pi$ in (A.4), we find that the value of $J(z, \xi)$ at $z = 0$ is given by

$$J(0, \xi) = i\xi^{\alpha-1} \int_{-\pi}^\pi (1 + e^{i\varphi})^{\frac{3\alpha}{2}-2} e^{i\varphi(1-\frac{\alpha}{2})} d\varphi, \quad \xi > 0.$$

It follows that

$$-\operatorname{Re} \int_{S_r} \mathcal{M}(z, \xi) dz = -\left(\int_{-\pi}^\pi (1 + e^{i\varphi})^{\frac{3\alpha}{2}-2} e^{i\varphi(1-\frac{\alpha}{2})} d\varphi \right) \left(\int_0^\pi \sin^{\alpha-2} \varphi d\varphi \right).$$

Since

$$\int_{-\pi}^\pi (1 + e^{i\varphi})^{\frac{3\alpha}{2}-2} e^{i\varphi(1-\frac{\alpha}{2})} d\varphi = \frac{2\pi\Gamma\left(\frac{3\alpha}{2}-1\right)}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma(\alpha)}, \quad \int_0^\pi \sin^{\alpha-2} \varphi d\varphi = \frac{\sqrt{\pi}\Gamma\left(\frac{\alpha-1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)},$$

equation (A.32) follows. \square

Remark A.11. The proof of Lemma A.10 shows that the constant c_α can be alternatively expressed as

$$c_\alpha = \int_{-\infty}^\infty \operatorname{Re} \mathcal{M}(x, y, \xi) dx,$$

where the right-hand side is independent of the choice of $y > 0$ and $\xi > 0$.

Lemma A.12. *We have*

$$|P(z, \xi)| \leq C (\arg z)^{\alpha-1}, \quad z \in \mathbb{H}, \quad \xi > 0, \quad (\text{A.33a})$$

$$|P(z, \xi) - 1| \leq C (\pi - \arg z)^{\alpha-1}, \quad z \in \mathbb{H}, \quad \xi > 0. \quad (\text{A.33b})$$

Proof. Using that $P(x, \xi) = 0$ for $x > 0$, we can write

$$P(re^{i\theta}, \xi) = -\frac{r}{c_\alpha} \operatorname{Re} \int_0^\theta \mathcal{M}(re^{i\varphi}, \xi) ie^{i\varphi} d\varphi, \quad r > 0, \quad \theta \in (0, \pi), \quad \xi > 0.$$

The estimate (A.19a) now yields

$$\begin{aligned} |P(re^{i\theta}, \xi)| &\leq Cr \int_0^\theta |\mathcal{M}(re^{i\varphi}, \xi)| d\varphi \leq C \int_0^\theta (\sin^{\alpha-2} \varphi) d\varphi \\ &\leq C \theta^{\alpha-1}, \end{aligned} \quad r > 0, \quad \theta \in (0, \pi), \quad \xi > 0,$$

which is (A.33a). Similarly, since $P(x, \xi) = 1$ for $x < 0$, we can write

$$P(re^{i\theta}, \xi) = 1 + \frac{r}{c_\alpha} \operatorname{Re} \int_\theta^\pi \mathcal{M}(re^{i\varphi}, \xi) ie^{i\varphi} d\varphi, \quad r > 0, \quad \theta \in (0, \pi), \quad \xi > 0.$$

The estimate (A.19a) now yields

$$\begin{aligned} |P(re^{i\theta}, \xi) - 1| &\leq Cr \int_\theta^\pi |\mathcal{M}(re^{i\varphi}, \xi)| d\varphi \leq C \int_\theta^\pi (\sin^{\alpha-2} \varphi) d\varphi \\ &\leq C(\pi - \theta)^{\alpha-1}, \end{aligned} \quad r > 0, \quad \theta \in (0, \pi), \quad \xi > 0,$$

which is (A.33b). □

Consider a system of two commuting SLEs in \mathbb{H} started from 0 and $\xi > 0$, respectively, with sufficiently regular growth speeds $\lambda_j(t) \geq 0$. Write ξ_t^1 and ξ_t^2 for the Loewner driving terms of the system and let g_t denote the solution of (2.1) which uniformizes the system at capacity t . Given $z \in \mathbb{H}$, let $Z_t = g_t(z)$ and let $\tau(z)$ denote the time that z is swallowed by the system.

Lemma A.13. *Let $z \in \mathbb{H}$. Define $P_t(z)$ by*

$$P_t(z) = P(Z_t - \xi_t^1, \xi_t^2 - \xi_t^1), \quad 0 \leq t < \tau_z.$$

Then $P_t(z)$ is a martingale for the system of commuting SLEs for any choice of the growth speeds $\lambda_j(t) \geq 0$.

Proof. Using the expression (A.28) for P , we have

$$P(Z_t - \xi_t^1, \xi_t^2 - \xi_t^1) = \frac{1}{c_\alpha} \int_{Z_t - \xi_t^1}^\infty \operatorname{Re} [\mathcal{M}(U, \xi_t^2 - \xi_t^1) dU].$$

Performing the change of variables $u = g_t^{-1}(U + \xi_t^1)$, this becomes

$$\begin{aligned} P_t(z) &= \frac{1}{c_\alpha} \int_z^\infty \operatorname{Re} [\mathcal{M}(U_t - \xi_t^1, \xi_t^2 - \xi_t^1) g'_t(u) du] \\ &= \frac{1}{c_\alpha} \int_z^\infty \operatorname{Re} [(\operatorname{Im} U_t)^{\alpha-2} (U_t - \xi_t^1)^{-\frac{\alpha}{2}} (U_t - \xi_t^2)^{-\frac{\alpha}{2}} (\bar{U}_t - \xi_t^1)^{1-\frac{\alpha}{2}} (\bar{U}_t - \xi_t^2)^{1-\frac{\alpha}{2}} \\ &\quad \times J(U_t - \xi_t^1, \xi_t^2 - \xi_t^1) g'_t(u) du], \quad 0 \leq t < \tau_z, \end{aligned}$$

where $U_t := g_t(u)$. Now

$$J(U_t - \xi_t^1, \xi_t^2 - \xi_t^1) = \int_{\bar{U}_t - \xi_t^1}^{U_t - \xi_t^1} (V - U_t + \xi_t^1)^\alpha (V - \bar{U}_t + \xi_t^1)^{\alpha-2} V^{-\frac{\alpha}{2}} (V - \xi_t^2 + \xi_t^1)^{-\frac{\alpha}{2}} dV,$$

and the change of variables $w = g_t^{-1}(V + \xi_t^1)$ gives

$$J(U_t - \xi_t^1, \xi_t^2 - \xi_t^1) = \int_{\bar{u}}^u (W_t - U_t)^\alpha (W_t - \bar{U}_t)^{\alpha-2} (W_t - \xi_t^1)^{-\frac{\alpha}{2}} (W_t - \xi_t^2)^{-\frac{\alpha}{2}} g'_t(w) dw,$$

where $W_t := g_t(W)$. Hence

$$P_t(z) = \frac{1}{c_\alpha} \int_z^\infty \operatorname{Re} \left[\int_{\bar{u}}^u Q dw du \right], \quad 0 \leq t < \tau_z,$$

where we use the short-hand notation Q for the integrand which is given by

$$Q := (\operatorname{Im} U_t)^{\alpha-2} (U_t - \xi_t^1)^{-\frac{\alpha}{2}} (U_t - \xi_t^2)^{-\frac{\alpha}{2}} (\bar{U}_t - \xi_t^1)^{1-\frac{\alpha}{2}} (\bar{U}_t - \xi_t^2)^{1-\frac{\alpha}{2}} \\ \times (W_t - U_t)^\alpha (W_t - \bar{U}_t)^{\alpha-2} (W_t - \xi_t^1)^{-\frac{\alpha}{2}} (W_t - \xi_t^2)^{-\frac{\alpha}{2}} g'_t(w) g'_t(u).$$

Since we have already established smoothness of P , we can apply Itô's formula to see that

$$dP_t(z) = \frac{1}{c_\alpha} \int_z^\infty \operatorname{Re} \left[\int_{\bar{u}}^u dQ dw du \right],$$

where

$$dQ = \frac{\partial Q}{\partial U_t} dU_t + \frac{\partial Q}{\partial \bar{U}_t} d\bar{U}_t + \frac{\partial Q}{\partial W_t} dW_t + \frac{\partial Q}{\partial g'_t(u)} dg'_t(u) + \frac{\partial Q}{\partial \overline{g'_t(u)}} d\overline{g'_t(u)} + \frac{\partial Q}{\partial g'_t(w)} dg'_t(w) \\ + \frac{\partial Q}{\partial \xi_t^1} d\xi_t^1 + \frac{\partial Q}{\partial \xi_t^2} d\xi_t^2 + \frac{\kappa}{4} \left(\lambda_1(t) \frac{\partial^2 Q}{(\partial \xi^1)^2} + \lambda_2(t) \frac{\partial^2 Q}{(\partial \xi^2)^2} \right). \quad (\text{A.34})$$

Substituting the expressions

$$dU_t = \sum_{j=1}^2 \frac{\lambda_j(t)}{U_t - \xi_t^j} dt, \quad d\bar{U}_t = \sum_{j=1}^2 \frac{\lambda_j(t)}{\bar{U}_t - \xi_t^j} dt, \quad dW_t = \sum_{j=1}^2 \frac{\lambda_j(t)}{W_t - \xi_t^j} dt, \\ d\xi_t^1 = \frac{\lambda_1(t) + \lambda_2(t)}{\xi_t^1 - \xi_t^2} dt + \sqrt{\frac{\kappa}{2} \lambda_1(t)} dB_t^1, \quad d\xi_t^2 = \frac{\lambda_1(t) + \lambda_2(t)}{\xi_t^2 - \xi_t^1} dt + \sqrt{\frac{\kappa}{2} \lambda_2(t)} dB_t^2, \\ dg'_t(u) = -g'_t(u) \sum_{j=1}^2 \frac{\lambda_j(t)}{(U_t - \xi_t^j)^2} dt, \quad d\overline{g'_t(u)} = -\overline{g'_t(u)} \sum_{j=1}^2 \frac{\lambda_j(t)}{(\bar{U}_t - \xi_t^j)^2} dt, \\ dg'_t(w) = -g'_t(w) \sum_{j=1}^2 \frac{\lambda_j(t)}{(W_t - \xi_t^j)^2} dt,$$

into (A.34), a long but straightforward computation shows that the drift term in dQ vanishes. This shows that $P_t(z)$ is a local martingale and since $P(z, \xi)$ is bounded it is actually a martingale. \square

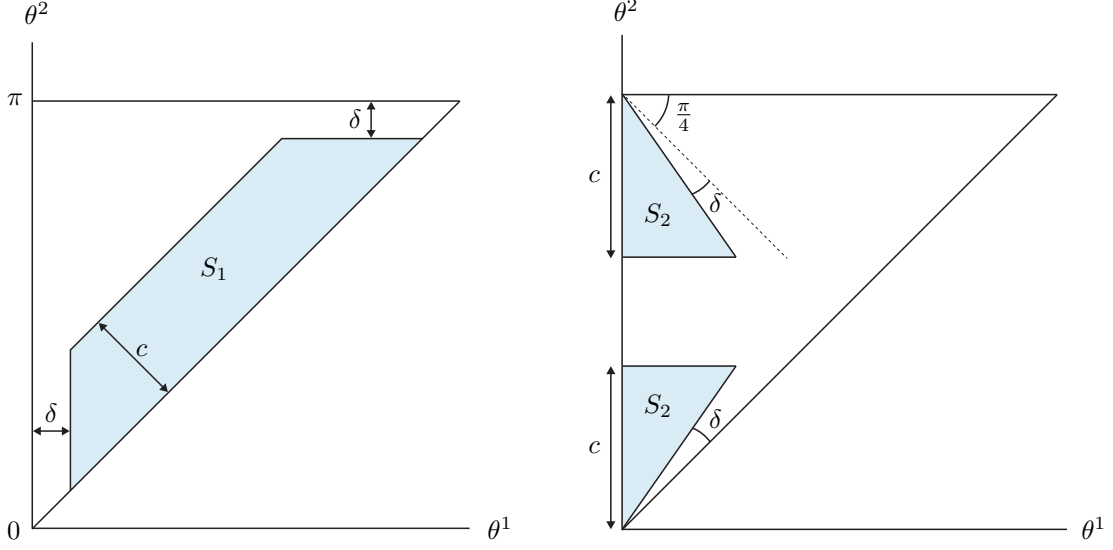


Figure 14. The asymptotic sectors S_1 and S_2 .

B Estimates for Green's function

The purpose of this appendix is to prove Lemma 6.2. By applying the method developed in Section 10, we can determine the asymptotic behavior of $h(\theta^1, \theta^2)$ near the boundary of Δ . This will prove Lemma 6.2.

B.1 The asymptotic sectors

Given $\delta > 0$ and $c > 0$, we define the open subsets S_j , $j = 1, \dots, 5$, of Δ by (see Figures 14, 15, and 16)

$$\begin{aligned}
 S_1 &= \{(\theta^1, \theta^2) \in \Delta \mid 0 < \theta^2 - \theta^1 < c\sqrt{2}, \theta^1 > \delta, \theta^2 < \pi - \delta\}, \\
 S_2 &= \left\{(\theta^1, \theta^2) \in \Delta \mid \theta^2 < c, \arctan \frac{\theta^1}{\theta^2} < \frac{\pi}{4} - \delta\right\} \\
 &\quad \cup \left\{(\theta^1, \theta^2) \in \Delta \mid \theta^2 > \pi - c, \arctan \frac{\theta^1}{\pi - \theta^2} < \frac{\pi}{4} - \delta\right\}, \\
 S_3 &= \left\{(\theta^1, \theta^2) \in \Delta \mid \theta^2 > \pi - c, \arctan \frac{\pi - \theta^2}{\theta^1} < \frac{\pi}{4} - \delta, \arctan \frac{\pi - \theta^2}{\pi - \theta^1} < \frac{\pi}{4} - \delta\right\}, \\
 S_4 &= \left\{(\theta^1, \theta^2) \in \Delta \mid \theta^1 + \theta^2 < c\sqrt{2}, \delta < \arctan \frac{\theta^1}{\theta^2} < \frac{\pi}{4}\right\} \\
 &\quad \cup \left\{(\theta^1, \theta^2) \in \Delta \mid \theta^2 - \theta^1 > \pi - c\sqrt{2}, \delta < \arctan \frac{\pi - \theta^2}{\theta^1} < \frac{\pi}{2} - \delta\right\} \\
 &\quad \cup \left\{(\theta^1, \theta^2) \in \Delta \mid \theta^1 + \theta^2 > 2\pi - c\sqrt{2}, \delta < \arctan \frac{\pi - \theta^2}{\pi - \theta^1} < \frac{\pi}{4}\right\}, \\
 S_5 &= \left\{(\theta^1, \theta^2) \in \Delta \mid \theta^1 < c, \theta^2 - \theta^1 > \delta, \theta^1 + \theta^2 < \pi - \delta\right\}.
 \end{aligned}$$

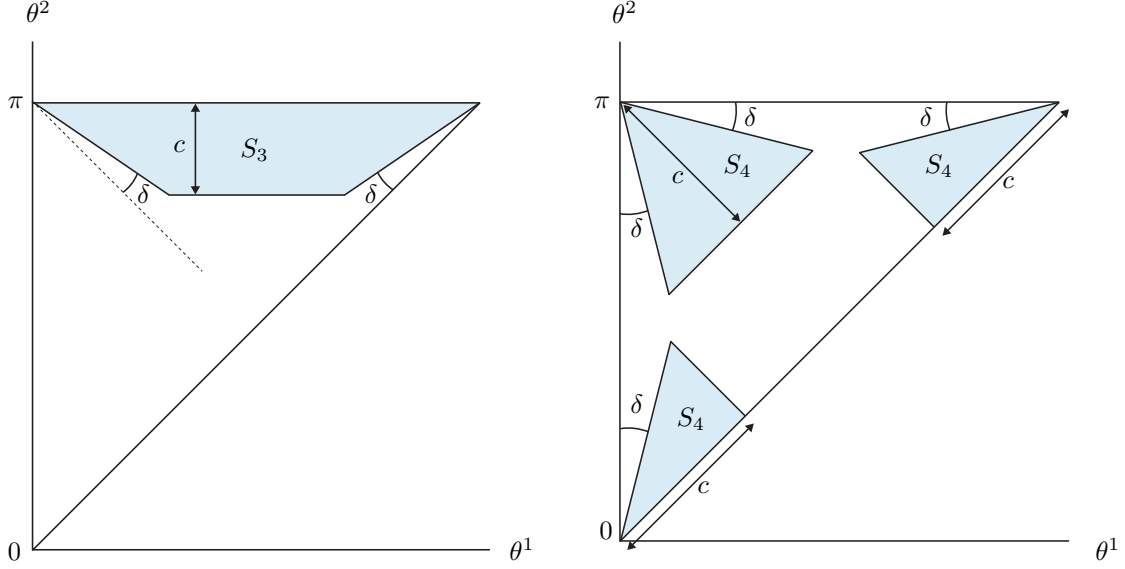


Figure 15. The asymptotic sectors S_3 and S_4 .

The asymptotics of $h(\theta^1, \theta^2)$ as (θ^1, θ^2) approaches the boundary of the triangle Δ , can be described in terms of the five asymptotic sectors $\{S_j\}_1^5$. Indeed, the first four sectors S_j , $j = 1, \dots, 4$, correspond to the different asymptotic regions studied in Sections 10.3-10.6 of Section 10, respectively, while the sector S_5 corresponds to the region where $w_2 \rightarrow \infty$.

Lemma B.1. *Let $\delta > 0$. Then there exist constants $c > 0$ and $\epsilon > 0$ such that the following estimates hold:*

- (1) $\text{dist}(w_1, \{0, 1, \infty\}) > \epsilon$ and $|w_2 - 1| < 1 - \epsilon$ for all $(\theta^1, \theta^2) \in S_1$,
- (2) $|w_1| < 1 - \epsilon$ and $|w_2| > 1 + \epsilon$ for all $(\theta^1, \theta^2) \in S_2$,
- (3) $|w_1| < 1 - \epsilon$ and $|w_2| < 1 - \epsilon$ for all $(\theta^1, \theta^2) \in S_3$,
- (4) $|w_1| < 1 - \epsilon$ and $\text{dist}(w_2, \{0, \infty\}) > \epsilon$ for all $(\theta^1, \theta^2) \in S_4$,
- (5) $\text{dist}(w_1, \{0, 1, \infty\}) > \epsilon$ and $|w_2| > 1 + \epsilon$ for all $(\theta^1, \theta^2) \in S_5$.

Proof. The proof follows easily from the definition (9.3) of w_1 and w_2 . □

B.2 Representations for h

The following lemma provides four representations for h which are suitable for determining the behavior of h for $(\theta^1, \theta^2) \in S_j$, $j = 1, \dots, 4$, respectively. For $(\theta^1, \theta^2) \in S_5$, we will use the representation (9.2) instead.

Lemma B.2. *Suppose $\alpha \geq 2$ satisfies $\frac{3\alpha}{2}, 2\alpha \notin \mathbb{Z}$. Then, for all $(\theta^1, \theta^2) \in \Delta$,*

$$h(\theta^1, \theta^2) = \frac{1}{\hat{c}} \sin^{\alpha-1}(\theta^1) \text{Im} \left[\sigma(\theta^2) (-e^{i\theta^2})^{\alpha-1} e^{-\frac{i\pi\alpha}{2}} P_1(w_1, w_2) \right], \quad (\text{B.1})$$

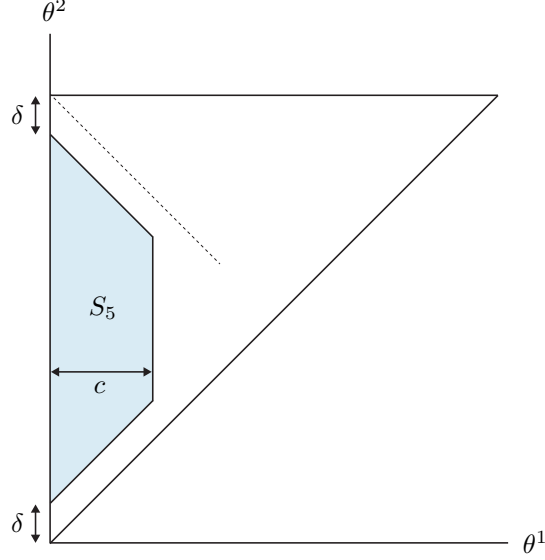


Figure 16. The asymptotic sector S_5 .

$$h(\theta^1, \theta^2) = \frac{1}{\hat{c}} \sin^{\alpha-1}(\theta^1) \operatorname{Im} \left[\sigma(\theta^2) (-e^{i\theta^2})^{\alpha-1} (Q_1(w_1, w_2) + w_1^{2\alpha-1} Q_2(w_1, w_2)) \right], \quad (\text{B.2})$$

$$h(\theta^1, \theta^2) = \frac{1}{\hat{c}} \sin^{\alpha-1}(\theta^1) \operatorname{Im} \left[\sigma(\theta_2) (-e^{i\theta^2})^{\alpha-1} (R_1(w_1, w_2) + w_2^{\frac{\alpha}{2}} R_2(w_1, w_2) + w_1^{2\alpha-1} Q_2(w_1, w_2)) \right], \quad (\text{B.3})$$

$$h(\theta^1, \theta^2) = \frac{1}{\hat{c}} \sin^{\alpha-1}(\theta^1) \operatorname{Im} \left[\sigma(\theta_2) (-e^{i\theta^2})^{\alpha-1} (e^{-\frac{i\pi\alpha}{2}} T_1(w_1, w_2) + w_1^{2\alpha-1} Q_2(w_1, w_2)) \right], \quad (\text{B.4})$$

where w_1, w_2 are given by (9.3).

Proof. Equations (B.2) and (B.3) follow from (9.2) together with the expressions (10.22) and (10.26) for F , respectively. Moreover, suppose we can show that

$$\operatorname{Im} X(\theta^1, \theta^2) = 0 \quad \text{in } \Delta, \quad (\text{B.5})$$

where

$$X(\theta^1, \theta^2) = \sigma(\theta^2) (-e^{i\theta^2})^{\alpha-1} e^{-\frac{i\pi\alpha}{2}} (w_2 - 1)^{1-\alpha} P_2(w_1, w_2), \quad (\text{B.6})$$

and the variables w_1 and w_2 in (B.6) are given by (9.3). Then equations (B.1) and (B.4) follow from (9.2) together with the expressions (10.17) and (10.31) for \tilde{F} and $F = \rho\tilde{F}$, respectively. It therefore only remains to show (B.5).

From the definition (10.16) of P_2 we see that for $w_1 \in \mathbb{C} \setminus [0, \infty)$ and $w_2 \in (0, 1)$ we have

$$P_2(w_1, w_2 + i0) = \frac{e^{i\pi(a+d)} \sin(a\pi)}{\sin(\pi(d+c))} e^{-i\pi(c+d+1)} |1 - w_2|^{a+b}$$

$$\times \int_A^{(0+,1+,0-,1-)} \left(s - 1 - \frac{1}{w_2 - 1}\right)^a \left(s - 1 + \frac{w_1 - 1}{w_2 - 1}\right)^b s^c (1 - s)^d ds, \quad (\text{B.7})$$

where a, b, c, d are given by (10.4). By definition, the value of $P_2(w_1, w_2)$ at a general point $(w_1, w_2) \in \mathcal{D}_1$ is determined by analytic continuation of (B.7) within the connected set $\mathcal{D}_1 \subset \mathbb{C}^2$. The branches of the complex powers in (B.7) are fixed by requiring that the principal branch is used initially at $s = A$. This means that whenever the points

$$A - 1 - \frac{1}{w_2 - 1} \quad \text{and} \quad A - 1 + \frac{w_1 - 1}{w_2 - 1}, \quad (\text{B.8})$$

cross the negative real axis during the analytic continuation, extra factors of $e^{\pm 2\pi ia}$ and $e^{\pm 2\pi ib}$, respectively, have to be inserted in (B.7).

In order to evaluate the function X in (B.6), we need the value of P_2 at points $(w_1, w_2) \in \mathcal{E}$, where \mathcal{E} denotes the subset of \mathbb{C}^2 characterized by (9.3), i.e.,

$$\mathcal{E} = \left\{ (w_1, w_2) = \left(1 - e^{-2i\theta^2}, \frac{\sin \theta^2}{\sin \theta^1} e^{-i(\theta^2 - \theta^1)} \right) \mid (\theta^1, \theta^2) \subset \Delta \right\}.$$

If w_1 and w_2 are given by (9.3), then

$$\cot \theta^2 < \cot \theta^1, \quad \frac{1}{w_2 - 1} = \frac{\cot \theta^2 + i}{\cot \theta^1 - \cot \theta^2}, \quad -\frac{w_1 - 1}{w_2 - 1} = \frac{\cot \theta^2 - i}{\cot \theta^1 - \cot \theta^2}.$$

Hence, we have, for all $(w_1, w_2) \in \mathcal{E}$,

$$\text{Im} \left(A - 1 - \frac{1}{w_2 - 1} \right) < 0, \quad \text{Im} \left(A - 1 + \frac{w_1 - 1}{w_2 - 1} \right) > 0. \quad (\text{B.9})$$

This shows that neither of the points in (B.8) crosses the negative real axis as long as (w_1, w_2) remains within \mathcal{E} . We can therefore find a formula for P_2 valid in \mathcal{E} as follows.

Let (w_1, w_2) be a point in \mathcal{E} corresponding to (θ^1, θ^2) via (9.3). Then

$$w_2 = 1 + \frac{\sin(\theta^2 - \theta^1)}{\sin \theta^1} e^{-i\theta^2}. \quad (\text{B.10})$$

Let $0 < \epsilon < |w_1 - 1|$ be small and let $(\tilde{w}_1(t), \tilde{w}_2(t))$, $t \in [0, 1]$, be the path in \mathcal{D}_1 defined by $\tilde{w}_1(t) = w_1$ for all t , while the path $\tilde{w}_2(t)$ starts at $1 - \epsilon + i0$, proceeds clockwise around the small circle of radius ϵ centered at 1 until it reaches the point $1 + \epsilon e^{-i\theta^2}$, and then proceeds along the segment $[1 + \epsilon e^{-i\theta^2}, w_2]$ until it reaches w_2 .

As \tilde{w}_2 moves along the arc from $1 - \epsilon + i0$ to $1 + \epsilon e^{-i\theta^2}$, the point $A - 1 - \frac{1}{\tilde{w}_2(t) - 1}$ crosses the negative real axis from the upper into the lower half-plane once (this adds a factor of $e^{2\pi ia}$ to (B.7)), and, provided that $\text{Im } w_1 \leq 0$ (i.e. $\theta^2 \geq \pi/2$), $A - 1 + \frac{\tilde{w}_1(t) - 1}{\tilde{w}_2(t) - 1}$ also crosses the negative real axis from the upper into the lower half-plane once (this adds a factor of $e^{2\pi ib}$ to (B.7)). If $\text{Im } w_1 > 0$, then $A - 1 + \frac{\tilde{w}_1(t) - 1}{\tilde{w}_2(t) - 1}$ does not cross the negative real axis. By varying θ^1 in (B.10), we see that the part of the path for which \tilde{w}_2 belongs to the segment $[1 + \epsilon e^{-i\theta^2}, w_2]$ lies in \mathcal{E} ; hence the analytic continuation along this part adds no more factors to (B.7). We end up with the following formula for P_2 in \mathcal{E} :

$$P_2(w_1, w_2) = \frac{e^{i\pi(a+d)} \sin(a\pi)}{\sin(\pi(d+c))} e^{-i\pi(a+b+c+d+1)} (w_2 - 1)^{a+b} e^{2\pi ia}$$

$$\begin{aligned} & \times \int_A^{(0+,1+,0-,1-)} \left(s-1-\frac{1}{w_2-1}\right)^a \left(s-1+\frac{w_1-1}{w_2-1}\right)^b s^c (1-s)^d ds \\ & \times \begin{cases} e^{2\pi ib}, & \text{Im } w_1 \leq 0, \\ 1, & \text{Im } w_1 > 0, \end{cases} \quad (w_1, w_2) \in \mathcal{E}. \end{aligned}$$

Substituting this formula into (B.6) and simplifying, we find

$$\begin{aligned} X(\theta^2, \theta^2) &= (-e^{i\theta^2})^{\alpha-1} (w_2-1)^{\alpha-1} e^{2\pi i\alpha} \\ &\times \int_A^{(0+,1+,0-,1-)} \left(s-1-\frac{1}{w_2-1}\right)^{\alpha-1} \left(s-1+\frac{w_1-1}{w_2-1}\right)^{\alpha-1} s^{-\frac{\alpha}{2}} (1-s)^{-\frac{\alpha}{2}} ds, \end{aligned}$$

where w_1, w_2 are given by (9.3). But

$$(-e^{i\theta^2})^{\alpha-1} (w_2-1)^{\alpha-1} = (\sin^{\alpha-1} \theta^2) (\cot \theta^1 - \cot \theta^2)^{\alpha-1} e^{-\pi i(\alpha-1)},$$

and, by (B.9),

$$\left(s-1-\frac{1}{w_2-1}\right)^{\alpha-1} \left(s-1+\frac{w_1-1}{w_2-1}\right)^{\alpha-1} = \left(\frac{1+((1-s)\cot \theta^1 + s\cot \theta^2)^2}{(\cot \theta^1 - \cot \theta^2)^2}\right)^{\alpha-1}.$$

Hence

$$\begin{aligned} X(\theta^2, \theta^2) &= -\sin^{\alpha-1}(\theta^2) (\cot \theta^2 - \cot \theta^1)^{\alpha-1} e^{\pi i\alpha} \\ &\times \int_A^{(0+,1+,0-,1-)} \left(\frac{1+((1-s)\cot \theta^1 + s\cot \theta^2)^2}{(\cot \theta^1 - \cot \theta^2)^2}\right)^{\alpha-1} s^{-\frac{\alpha}{2}} (1-s)^{-\frac{\alpha}{2}} ds. \end{aligned} \tag{B.11}$$

If $g(s)$ is an analytic function, the identity (9.16) implies

$$\begin{aligned} & \overline{\int_A^{(0+,1+,0-,1-)} g(s) s^c (1-s)^d ds} = \int_A^{(0-,1-,0+,1+)} \overline{g(\bar{s})} s^c (1-s)^d ds \\ &= \frac{-1 + e^{-2\pi ic} - e^{-2\pi i(c+d)} + e^{-2\pi id}}{-1 + e^{2\pi ic} - e^{2\pi i(c+d)} + e^{2\pi id}} \int_A^{(0+,1+,0-,1-)} \overline{g(\bar{s})} s^c (1-s)^d ds \\ &= e^{-2\pi i(c+d)} \int_A^{(0+,1+,0-,1-)} \overline{g(\bar{s})} s^c (1-s)^d ds. \end{aligned}$$

Using this identity to compute the imaginary part of (B.11) we arrive at

$$\begin{aligned} \text{Im } X(\theta^1, \theta^2) &= -\sin^{\alpha-1}(\theta^2) (\cot \theta^2 - \cot \theta^1)^{\alpha-1} \int_A^{(0+,1+,0-,1-)} (e^{\pi i\alpha} - e^{-\pi i\alpha} e^{-2i\pi(-\frac{\alpha}{2}-\frac{\alpha}{2})}) \\ &\times \left(\frac{1+((1-s)\cot \theta^1 + s\cot \theta^2)^2}{(\cot \theta^1 - \cot \theta^2)^2}\right)^{\alpha-1} s^{-\frac{\alpha}{2}} (1-s)^{-\frac{\alpha}{2}} ds = 0. \end{aligned}$$

This proves (B.5) and completes the proof of the lemma. \square

B.3 Proof of Lemma 6.2

We are now in a position to prove Lemma 6.2. Indeed, since h clearly is smooth in the interior of Δ and the parameter $\delta > 0$ which defines the sectors S_j is arbitrary, Lemma 6.2 is a direct consequence of the following result.

Lemma B.3. *Let $\alpha \geq 2$. Then the function $h(\theta^1, \theta^2)$ defined in (6.11) satisfies the following estimates:*

$$|h(\theta^1, \theta^2)| \leq C \sin^{\alpha-1} \theta^1, \quad (\theta^1, \theta^2) \in \cup_{j=1}^5 S_j, \quad (\text{B.12})$$

$$\frac{|h(\theta^1, \theta^2) - h_f(\theta^2)|}{\sin^{\alpha-1} \theta^1} \leq C \frac{|\theta^2 - \theta^1|}{\sin \theta^1}, \quad (\theta^1, \theta^2) \in S_1, \quad (\text{B.13})$$

$$\frac{|h(\theta^1, \theta^2) - \sin^{\alpha-1} \theta^1|}{\sin^{\alpha-1} \theta^1} \leq C \frac{\sin \theta^2}{\sin \theta^1}, \quad (\theta^1, \theta^2) \in S_3, \quad (\text{B.14})$$

where $h_f(\theta)$ is defined in (8.8).

Proof. Let us first assume that $\alpha \geq 2$ satisfies $\frac{3\alpha}{2}, 2\alpha \notin \mathbb{Z}$. Equation (B.1), Lemma B.1 (1), and Lemma 10.5 (a) show that (B.12) holds in S_1 . Also, by Lemma 10.5 (a), the following estimate is valid in S_1 :

$$\frac{|h(\theta^1, \theta^2) - h(\theta^2, \theta^2)|}{\sin^{\alpha-1} \theta^1} \leq C |w_2 - 1| = C \frac{\sin(\theta^2 - \theta^1)}{\sin \theta^1} \leq C \frac{|\theta^2 - \theta^1|}{\sin \theta^1}, \quad (\text{B.15})$$

where

$$h(\theta^2, \theta^2) = \frac{1}{\hat{c}} \sin^{\alpha-1}(\theta^2) \text{Im} \left[\sigma(\theta^2) (-e^{i\theta^2})^{\alpha-1} e^{-\frac{i\pi\alpha}{2}} P_1(w_1, 1) \right].$$

For a, b, c, d given by (10.4), we have (cf. (8.13))

$$\begin{aligned} P_1(w_1, 1) &= \frac{e^{-i\pi c} \sin(d\pi)}{\sin(\pi(d+c))} \int_A^{(0+, 1+, 0-, 1-)} v^a (v - w_1)^b (1 - v)^{c+d} dv \\ &= 2i\pi (-1 + e^{i\pi\alpha}) (-w_1)^{\alpha-1} {}_2F_1 \left(1 - \alpha, \alpha; 1; \frac{1}{w_1} \right). \end{aligned}$$

It follows that $h(\theta^1, \theta^1) = h_f(\theta^1)$ where h_f is given by (8.8). Equation (B.13) then follows from (B.15).

Using the fact that $\text{dist}(\frac{w_2}{w_1}, \{0, 1\}) > \epsilon$ for all $(\theta^1, \theta^2) \in \Delta$, Lemma B.1 (2) and Lemma 10.5 (b) and (c) imply

$$|Q_1 + w_1^{2\alpha-1} Q_2| \leq C |w_2|^{-\frac{\alpha}{2}} + C |w_1|^{\frac{3\alpha}{2}-1} \leq C, \quad (\theta^1, \theta^2) \in S_2.$$

Hence equation (B.2) shows that (B.12) holds in S_2 .

Similarly, Lemma B.1 (3), and Lemma 10.5 (c), (d), and (e) show that

$$|R_1 + w_2^{\frac{\alpha}{2}} R_2 + w_1^{2\alpha-1} Q_2| \leq C + C |w_2|^{\frac{3\alpha}{2}-1} + C |w_1|^{\frac{3\alpha}{2}-1} \leq C, \quad (\theta^1, \theta^2) \in S_3.$$

Hence equation (B.3) implies that (B.12) holds in S_3 . Also, by Lemma 10.5 (d), since $\alpha \geq 2$, the following estimate is valid in S_3 :

$$\frac{|h(\theta^1, \theta^2) - h(\theta^1, \pi)|}{\sin^{\alpha-1} \theta^1} \leq C(|w_1| + |w_2|) + C |w_2|^{\frac{3\alpha}{2}-1} + C |w_1|^{\frac{3\alpha}{2}-1} \leq C(|w_1| + |w_2|)$$

$$\leq C|\pi - \theta^2| + C\left|\frac{\sin \theta^2}{\sin \theta^1}\right| \leq C\left|\frac{\sin \theta^2}{\sin \theta^1}\right|,$$

where

$$h(\theta^1, \pi) = \frac{\sin^{\alpha-1} \theta^1}{\hat{c}} \operatorname{Im} [e^{-i\pi\alpha} R_1(0, 0)].$$

For a, b, c, d given by (10.4), we have

$$\begin{aligned} R_1(0, 0) &= \frac{e^{2\pi ia} - 1}{e^{2\pi i(a+b+c)} - 1} \int_A^{(0+, 1+, 0-, 1-)} v^{a+b+c} (1-v)^d dv \\ &= - \frac{(e^{2i\pi a} - 1)(e^{2i\pi d} - 1)\Gamma(d+1)\Gamma(a+b+c+1)}{\Gamma(a+b+c+d+2)} \\ &= - \frac{(e^{2i\pi\alpha} - 1)(e^{-i\pi\alpha} - 1)\Gamma(1 - \frac{\alpha}{2})\Gamma(\frac{3\alpha}{2} - 1)}{\Gamma(\alpha)}. \end{aligned}$$

Taking the definition (2.9) of \hat{c} into account, it follows that $h(\theta^1, \pi) = \sin^{\alpha-1} \theta^1$. This proves (B.14).

Lemma B.1 (4), and Lemma 10.5 (c) and (f) show that

$$|e^{-\frac{i\pi\alpha}{2}} T_1 + w_1^{2\alpha-1} Q_2| \leq C + C|w_1|^{-\frac{\alpha}{2}} \leq C, \quad (\theta^1, \theta^2) \in S_4.$$

Hence equation (B.4) implies that (B.12) holds in S_4 .

Lemma B.1 (5), and Lemma 10.5 (g) show that

$$|F| \leq C|w_2|^{\frac{3\alpha}{2}-1} \leq C, \quad (\theta^1, \theta^2) \in S_5.$$

Hence equation (9.2) shows that (B.12) holds in S_5 . This completes the proof of the lemma in the case when $\frac{3\alpha}{2}$ and 2α are not integers.

Assume finally that $\frac{3\alpha}{2}$ and/or 2α is an integer. Then some of the functions in Lemma 10.5 degenerate, so a slightly different argument is required. We do not give complete details, but outline the relevant steps.

Suppose first that $\alpha \notin \mathbb{Z}$ but $\frac{3\alpha}{2}$ or 2α is an integer. Then the limit $w_2 \rightarrow 1$ can still be treated as in Section 10, because $c + d = -\alpha$ is not an integer. However, the limits involving $w_1 \rightarrow 0$ or $w_2 \rightarrow 0$ cannot be treated in the same way in general, because $a + b = 2\alpha - 2$ and/or $a + b + c = \frac{3\alpha}{2} - 2$ is an integer. However, since $\alpha \geq 2$, we have $a + b > 0$ and $a + b + c > 0$. Hence the integral (9.1) defining F is nonsingular at $v = 0$ (also in the limit as w_1 and w_2 approach zero). Hence, we can derive the leading behavior of F in these regimes using the following alternative approach: First, we collapse the two loops of the Pochhammer contour enclosing the origin down to the interval $[0, A]$ (cf. equation (9.12)). Then we find the leading-order asymptotics by Taylor expanding the integrand as w_1 and/or w_2 approaches zero.

Assume finally that $\alpha = n \geq 2$ is an integer. This case is considered in Section 9, where an expression for $h(\theta^1, \theta^2; n)$ is derived by taking the limit of the defining equation (9.1) for F as $\alpha \rightarrow n$. In order to prove (B.12)-(B.14) in this case, we compute the limits as $\alpha \rightarrow n$ of each of the four equations in Lemma B.2. This gives four analogous equations valid for $\alpha = n$. As above, it follows from these equations that h satisfies (B.12)-(B.14). The crucial point is that the singular contribution from P_2 vanishes as a consequence of (B.5). \square

References

- [1] T. Alberts, I. Binder, and F. Viklund, A dimension spectrum for SLE boundary collisions, *Comm. Math. Phys.* **343** (2016), 273–298.
- [2] T. Alberts, M. J. Kozdron, and G. F. Lawler, The Green function for the radial Schramm-Loewner evolution, *J. Phys. A* **45** (2012), 494015, 17 pp.
- [3] T. Alberts, N.-G. Kang, and N. Makarov, *In preparation*.
- [4] D. Beliaev and F. Johansson Viklund, Some remarks on SLE bubbles and Schramm’s two-point observable, *Comm. Math. Phys.* **320** (2013), 379–394.
- [5] M. Bauer and D. Bernard, SLE_κ growth processes and conformal field theories, *Phys. Lett B* **543** (2002), 135–138.
- [6] M. Bauer and D. Bernard, Conformal field theories of stochastic Loewner evolutions, *Comm. Math. Phys.* **239** (2003), 493–521.
- [7] M. Bauer, D. Bernard, and K. Kytölä, Multiple Schramm-Loewner evolutions and statistical mechanics martingales, *J. Stat. Phys.* **120** (2005), 1125–1163.
- [8] A. A. Beliaev, A. M. Polyakov, and A. B. Zamolodchikov, Infinite conformal symmetry in two-dimensional quantum field theory, *Nucl. Phys. B* **241** (1984), 333–380.
- [9] D. Beliaev and S. Smirnov, Harmonic measure and SLE, *Comm. Math. Phys.* **290** (2009), 577–595.
- [10] J. Cardy, Conformal invariance and surface critical behavior, *Nucl. Phys. B* **240** (1984), 514–532.
- [11] J. Cardy and J. J. H. Simmons, Twist operator correlation functions in $O(n)$ loop models, *J. Phys. A: Mathematical and Theoretical* **42** (2009)
- [12] P. Di Francesco, P. Mathieu and D. Sénéchal, *Conformal Field Theory*, Graduate Texts in Contemporary Physics, Springer-Verlag, New York, 1997.
- [13] V. S. Dotsenko and V. A. Fateev, Conformal algebra and multipoint correlation functions in 2D statistical models, *Nuclear Phys. B* **240** (1984), 312–348.
- [14] J. Dubédat, Euler integrals for commuting SLEs, *J. Stat. Phys* **123** (2006), 1183–1218.
- [15] J. Dubédat, Commutation relations for Schramm-Loewner evolutions, *Comm. Pure Appl. Math.* **60** (2007), 1792–1847.
- [16] J. Dubédat, SLE and Virasoro representations: fusion, *Comm. Math. Phys.* **336** (2015), 761–809.
- [17] C. Ferreira and J. L. López, Asymptotic expansions of the Appell’s function F_1 , *Quart. Appl. Math.* **62** (2004), 235–257.
- [18] L. Field and G. F. Lawler, Escape probability and transience for SLE, *Electron. J. Probab.* **20** (2015).
- [19] S. M. Flores and P. Kleban, A solution space for a system of null-state partial differential equations: Part 1, *Comm. Math. Phys.* **333** (2015), 389–434.

- [20] A. Gamsa and J. Cardy, The scaling limit of two cluster boundaries in critical lattice models, *J. Stat. Mech. Theory Exp.* (2005).
- [21] J. B. Garnett and D. E. Marshall, *Harmonic measure*, New Mathematical Monographs 2, Cambridge University Press, Cambridge, 2005.
- [22] V. A. Golubeva and A. N. Ivanov, *Fuchsian systems for Dotsenko-Fateev multipoint correlation functions and similar integrals of hypergeometric type*, preprint, arXiv:1611.07758.
- [23] N.-G. Kang and N. G. Makarov, Gaussian free field and conformal field theory, *Astérisque* **353** (2013), viii+136 pp.
- [24] I. Karatzas and S. E. Shreve, *Brownian motion and stochastic calculus*, Second edition, Graduate Texts in Mathematics 113, Springer-Verlag, New York, 1991.
- [25] K. Kytölä and E. Peltola, Conformally covariant boundary correlation functions with a quantum group, (2016) Preprint.
- [26] K. Kytölä and E. Peltola, Pure partition functions of multiple SLEs, *Comm. Math. Phys.* **346** (2016), 237–292.
- [27] G. F. Lawler, *Conformally invariant processes in the plane*, Mathematical Surveys and Monographs, 114. American Mathematical Society, Providence, RI, 2005.
- [28] G. F. Lawler, Minkowski content of the intersection of a Schramm-Loewner evolution (SLE) curve with the real line, *J. Math. Soc. Japan* **67** (2015), 1631–1669.
- [29] G. F. Lawler, *Lecture notes on the Bessel process*, unpublished notes, available at <http://www.math.uchicago.edu/~lawler/bessel.pdf>.
- [30] G. F. Lawler and B. M. Werner, Multi-point Green’s functions for SLE and an estimate of Beffara, *Ann. Probab.* **41** (2013), 1513–1555.
- [31] J. Miller and S. Sheffield, Imaginary geometry I: Interacting SLEs, *Probab. Theory Related Fields* **164** (2016), 553–705.
- [32] J. Miller and H. Wu, Intersections of SLE paths: the double and cut point dimension of SLE, *Probab. Theory Related Fields* (2016), doi:10.1007/s00440-015-0677-x.
- [33] NIST Digital Library of Mathematical Functions, <http://dlmf.nist.gov/>, Release 1.0.13 of 2016-09-16. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, and B. V. Saunders, eds.
- [34] F. W. J. Olver, *Asymptotics and special functions*, Academic Press, New York, 1974.
- [35] O. Schramm, A percolation formula, *Electron. Comm. Probab.* **6** (2001), 115–120.